10/27/2017 Fall 2017, Calculus I Test II (Version I) Solutions Sitaraman

Howard University Math Department

Instructions:
PLEASE PROVIDE STEP BY STEP EXPLANATIONS
WRITING ONLY ANSWERS WILL NOT GET FULL CREDIT
Time Limit 50 minutes. Anything you get over 100 points is extra credit.
Please read the questions carefully before answering

1. (40 points]) Find the derivatives of the following functions:

   $(a)\ x^3 \sin x \qquad (b)\ e^{(1+\ln x)} \qquad (c)\ \dfrac{x+1}{x-1} \qquad (d)\ \cos^{-1} x + \tan x$

   Solution:

   (a) Use product rule to get $f'(x) = (\sin x)'x^3 + (\sin x)(x^3)'$
   $= (\cos x)x^3 + \sin x(3x^2) = x^2(x\cos x + 3\sin x).$

   (b) Use chain rule with $y = e^u,\ u = 1 + \ln x$ and $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$
   $= e^u(0 + (1/x)) = e^{(1+\ln x)}/x = (e^1 e^{\ln x})/x = (ex)/x = e.$

   You can also do this by simplifying first using properties of logarithms: $e^{(1+\ln x)} = e^1 e^{\ln x} = ex$ and so its derivative is just e because e is just a fixed number. (Only when you raise it to the power of x or some other variable you get a function).

   (c) Use quotient rule: $\left(\dfrac{x+1}{x-1}\right)' = \dfrac{(x+1)'(x-1) - (x+1)(x-1)'}{(x-1)^2}$
   $= \dfrac{(1)(x-1) - (x+1)(1)}{(x-1)^2} = \dfrac{-2}{(x-1)^2}$

   (d) $f'(x) = (cos^{-1}x)' + (\tan x)' = \dfrac{-1}{\sqrt{1-x^2}} + \sec^2 x$. You can find the derivative of $\tan x$ using quotient rule for sine/cosine or just write it as $sec^2 x$ if you remember it.

2. (20 points) Use implicit differentiation to find the equation to the tangent of the curve $y^2 = x^5 + x$ at $(1, 2)$.

   Solution:

   Differentiating both sides with respect to x we get $2yy' = 5x^4 + 1 \implies y' = \dfrac{5x^4+1}{2y}$.

   At $(1, 2)$ the slope m is given by putting $x = 1, y = 2$.

   We get $m = (5(1^4) + 1)/(2(2)) = 6/4 = 3/2$.

   So the equation of the tangent is $y - 2 = (3/2)(x - 1)$. This is $y = 1.5x + 0.5$ after simplification.

3. The population of a town after t years is given by $P(t) = 5000e^{-0.01t}$.

   a) (6 points) Find the rate of change of this population in the beginning. Is it increasing or decreasing?

   b)(4 points) Is it always increasing or decreasing?

   c) (5 points) When would it become half (i.e, 2500) ?

   [You may need that $\ln 2 = 0.693$].

   d) (5 points) What happens to the population as t approaches ∞ ?

   Solution:

a) Need to find $P'(0)$. Rate of change after 5 years is given by $P'(5)$. We have, using chain rule:

$$P'(t) = 5000(-0.1e^{-0.01t}) = -50e^{-0.01t}.$$

So $P'(0) = -50e^{-0.01(0)} = -50e^0 = -50$.

So it is decreasing at the rate of 50 people per year.

b) You can see from the formula for $P'(t)$ that the derivative is always negative. This is because the exponential function is always positive and we are multiplying it by -0.01. So the function is always decreasing.

c) Solving $(1/2)P(0) = P(0)e^{-0.01t}$ we get $t = \ln(1/2)/(-0.01) = \ln(2^{-1})/(-0.01) = (-1)(\ln 2)/(-0.01) =$ 69.3 years.

d) As $t \to \infty$ we have $e^{-0.01t} = 1/e^{0.01t} \to 0$. So $P(t) \to 0$.

4. The following concerns $f(x) = e^x - x$.

   (a) (8 points) Find all the critical points and inflexion points.

   (b) (8 points) Using derivatives show where function is increasing and decreasing, concave up and concave down, and has local maximum or minimum.

   (c) (4 points) Show that the absolute minimum of f *over the entire real line* is at $x = 0$ and find that value.

Solution:

a) The critical points are found by looking at x where $f'(x)$ is zero or undefined.

We have $f'(x) = e^x - 1$. This is well defined everywhere. So we only need to find the points where derivative is zero.

Setting $f'(x) = 0$ we get $e^x - 1 = 0 \implies x = 0$.

So 0 is the only critical point *over the entire real line.*

To find inflexion points we need to see where $f''(x) = 0$.

Setting $f''(x) = 0$ we get $e^x = 0$.

But $e^x > 0$ always, so there are no infexion points for this function anywhere.

b) Since $f''(x) > 0$ for all x, the function is always concave up.

Clearly $f'(x) = e^x - 1$ is negative when $x < 0$ and positive when $x > 0$. So function is decreasing for $x < 0$ and increasing when $x > 0$.

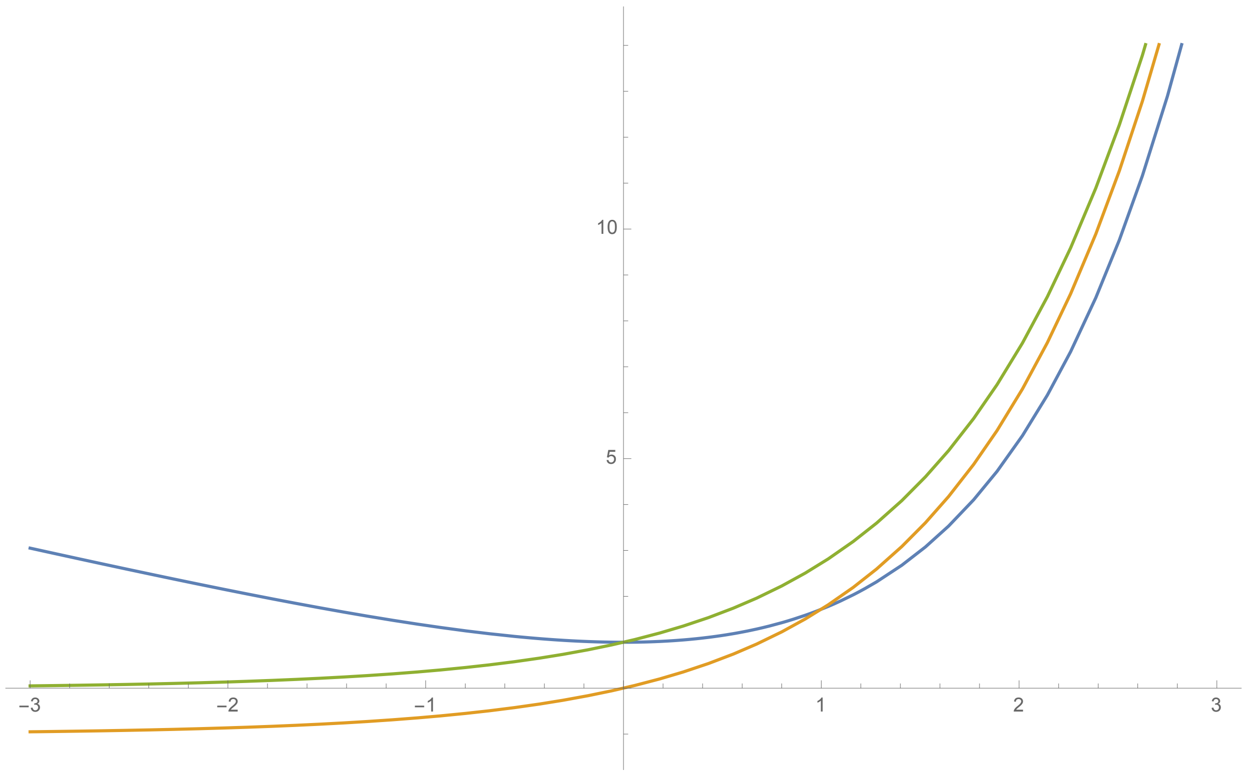
Also by first derivative test (decreasing before $x = 0$, increasing after) AND second derivative test ($f''(0) > 0$) at $x = 0$ function has local minimum.

c) At $x = 0$ function has absolute minimum because it is decreasing before that point, and increasing afterwards, and there are no other turning points. If there were other values below the value at 0 then function would have to turn around somewhere.

The minimum value is $f(0) = e^0 - (0) = 1$.

BTW there is no absolute maximum over the entire real line because the function goes to infinity on either side of $x-$axis. If you look in some closed interval it will have absolute max and min on the boundary points.

See graph below to understand all of this better. The blue line is the function, orange line its derivative and green line the second derivative.



5. (extra credit 20 points) A balloon in the shape of a sphere has radius $r = r(t)$ inches and volume $V(t) = \frac{4}{3}\pi r^3$ at time t seconds. Note that r is a function of time. State whether the following are true or false. If true explain why and if false disprove it or provide counterexample:

(a) The rate at which the volume is changing with respect to time is given by $4\pi r^2$ units per second.

(b) The change in volume when radius increases from 5 inches to 5 and a quarter inches is approximately 25π, the area of a circle of radius 5 inches. For this part of the question think of V as a function or r.

(c) The speed at which the radius is increasing at a given time is given by volume/ (surface area) at that time. Note that surface area of sphere is $4\pi r^2$.

(d) If the radius is increased by ten percent the diameter also increases by ten percent.

Soln:

(a) False. The rate of change of volume *with respect to time* is given by $V'(t) = (dV/dr)(dr/dt) = 4\pi r^2(dr/dt) = 4\pi r^2 r'(t)$.

(b) True. This can be done using differentials: $dV = V'(5)dr = 4\pi(5^2)(1/4) = 25\pi$.

(c) False. $V'(t) = V'(r)r'(t) \implies r'(t) = V'(t)/V'(r) = V'(t)/(4\pi r^2)$. So the numerator must be *rate of change of volume with respect to time* not volume itself.

(d) True. If $dr/r = 0.1$ then $d(2r)/(2r) = 2dr/(2r) = dr/r = 0.1$ also. In simple terms, if r goes from 10 to 11, diameter goes from 20 to 22, an increase of 2 which is 10 percent of 20.

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1. (40 points]) Find the derivatives of the following functions:

   $(a)\ x^2 \sin x \qquad (b)\ e^{(2+\ln x)} \qquad (c)\ \dfrac{x+1}{x-1} \qquad (d)\ \sin^{-1} x + \tan x$

   Solution:

   (a) Use product rule to get

   $(\cos x)x^2 + \sin x(2x) = x(x\cos x + 2\sin x)$.

   (b) Use chain rule with $y = e^u,\ u = 2 + \ln x$ and $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$

   $$= e^u(0 + (1/x)) = e^{(2+\ln x)}/x = (e^2 e^{\ln x})/x = (e^2 x)/x = e^2.$$

   You can also do this by simplifying first using properties of logarithms: $e^{(2+\ln x)} = e^2 e^{\ln x} = e^2 x$ and so its derivative is just e^2 because e is just a fixed number. (Only when you raise it to the power of x or some other variable you get a function).

   (c) Use quotient rule: $\left(\dfrac{x+1}{x-1}\right)' = \dfrac{(x+1)'(x-1) - (x+1)(x-1)'}{(x-1)^2}$

   $$= \frac{(1)(x-1) - (x+1)(1)}{(x-1)^2} = \frac{-2}{(x-1)^2}$$

   (d) $f'(x) = (\sin^{-1} x)' + (\tan x)' = \dfrac{1}{\sqrt{1-x^2}} + \sec^2 x$. You can find the derivative of $\tan x$ using quotient rule for sine/cosine or just write it as $sec^2 x$ if you remember it.

2. (20 points) Use implicit differentiation to find the equation to the tangent of the curve $y^2 = x^4 + x$ at $(1, 2)$.

   Solution:

   Differentiating both sides with respect to x we get $2yy' = 4x^3 + 1 \implies y' = \dfrac{4x^3+1}{2y}$.

   At $(1, 2)$ the slope m is given by putting $x = 1, y = 2$.

   We get $m = (4(1^3) + 1)/(2(2)) = 5/4$.

   So the equation of the tangent is $y - 2 = (5/4)(x - 1)$. This is $y = 1.25x + 0.75$ after simplification.

3. The population of a town after t years is given by $P(t) = 8000e^{-0.01t}$.

   a) (6 points) Find the rate of change of this population in the beginning. Is it increasing or decreasing?

   b)(4 points) Is it always increasing or decreasing?

   c) (5 points) When would it become half (i.e, 4000) ?

   [You may need that $\ln 2 = 0.693$].

   d) (5 points) What happens to the population as t approaches ∞ ?

   Solution:

a) Need to find $P'(0)$. Rate of change after 5 years is given by $P'(5)$. We have, using chain rule:

$$P'(t) = 8000(-0.1e^{-0.01t}) = -80e^{-0.01t}.$$

So $P'(0) = -80e^{-0.01(0)} = -80e^0 = -80$.

So it is decreasing at the rate of 80 people per year.

b) You can see from the formula for $P'(t)$ that the derivative is always negative. This is because the exponential function is always positive and we are multiplying it by -0.01. So the function is always decreasing.

c) Solving $(1/2)P(0) = P(0)e^{-0.01t}$ we get $t = \ln(1/2)/(-0.01) = \ln(2^{-1})/(-0.01) = (-1)(\ln 2)/(-0.01) =$ 69.3 years.

d) As $t \to \infty$ we have $e^{-0.01t} = 1/e^{0.01t} \to 0$. So $P(t) \to 0$.

4. The following concerns $f(x) = e^x - 2x$.

   (a) (8 points) Find all the critical points and inflexion points.

   (b) (8 points) Using derivatives show where function is increasing and decreasing, concave up and concave down, and has local maximum or minimum.

   (c) (4 points) Show that the absolute minimum of f *over the entire real line* is at $x = \ln 2$ and find that value. You can leave your answer in terms of $\ln 2$.

Solution:

a) The critical points are found by looking at x where $f'(x)$ is zero or undefined.

We have $f'(x) = e^x - 2$. This is well defined everywhere. So we only need to find the points where derivative is zero.

Setting $f'(x) = 0$ we get $e^x - 2 = 0 \implies x = \ln 2$.

So $\ln 2$ is the only critical point *over the entire real line.*

To find inflexion points we need to see where $f''(x) = 0$.

Setting $f''(x) = 0$ we get $e^x = 0$.

But $e^x > 0$ always, so there are no infexion points for this function anywhere.

b) Since $f''(x) > 0$ for all x, the function is always concave up.

Clearly $f'(x) = e^x - 2$ is negative when $x < \ln 2$ and positive when $x > \ln 2$. So function is decreasing for $x < \ln 2$ and increasing when $x > \ln 2$.

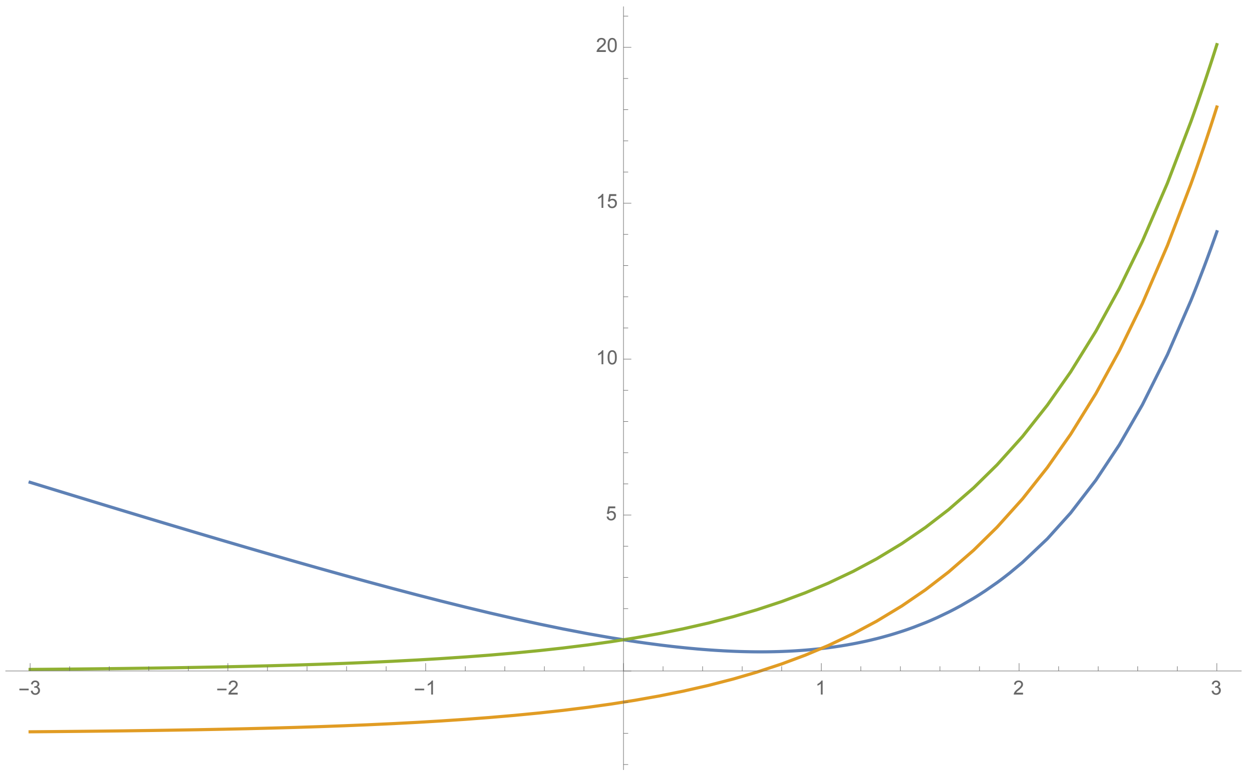
Also by first derivative test (decreasing before $x = \ln 2$, increasing after) AND second derivative test ($f''(\ln 2) > 0$) at $x = \ln 2$ function has local minimum.

c) At $x = \ln 2$ function has absolute minimum because it is decreasing before that point, and increasing afterwards, and there are no other turning points. If there were other values below the value at $\ln 2$ then function would have to turn around somewhere.

The minimum value is $f(\ln 2) = e^{\ln 2} - 2(\ln 2) = 2 - 2(\ln 2) = 2(1 - \ln 2) = 2(1 - 0.693) = 0.307$.

BTW there is no absolute maximum over the entire real line because the function goes to infinity on either side of $x-$axis. If you look in some closed interval it will have absolute max and min on the boundary points.

See graph below to understand all of this better. The blue line is the function, orange line its derivative and green line the second derivative.



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   (a) The rate at which the volume is changing with respect to time is given by $4\pi r^2$ units per second.

   (b) The change in volume when radius increases from 5 inches to 5 and a quarter inches is approximately 6.25π, the area of a circle of radius 2.5 inches. For this part of the question think of V as a function or r.

   (c) The speed at which the radius is increasing at a given time is given by (rate of change of volume with respect to time)/ (surface area) at that time. Note that surface area of sphere is $4\pi r^2$.

   (d) If the radius is increased by ten percent the diameter also increases by ten percent.

   Soln:

   (a) False. The rate of change of volume *with respect to time* is given by $V'(t) = (dV/dr)(dr/dt) = 4\pi r^2(dr/dt) = 4\pi r^2 r'(t)$.

   (b) False. This can be done using differentials: $dV = V'(5)dr = 4\pi(5^2)(1/4) = 25\pi$.

   (c) True. $V'(t) = V'(r)r'(t) \implies r'(t) = V'(t)/V'(r) = V'(t)/(4\pi r^2)$. So the numerator must be rate of change of volume.

   (d)True. If $dr/r = 0.1$ then $d(2r)/(2r) = 2dr/(2r) = dr/r = 0.1$ also. In simple terms, if r goes from 10 to 11, diameter goes from 20 to 22, an increase of 2 which is 10 percent of 20.