9/21/2017 Fall 2017, Calculus I Test I (Version I) Solutions Sitaraman

Howard University Math Department

Instructions:
PLEASE PROVIDE STEP BY STEP EXPLANATIONS
WRITING ONLY ANSWERS WILL NOT GET FULL CREDIT
Time Limit 50 minutes. Total 100 points. Each 20 points.
Please read the questions carefully before answering

1. Given $f(x) = \cos x$ and $g(x) = \frac{\pi}{x}$ find the following:

   (a) (6 points) $f \circ g(x)$ (b) (6 points) $g \circ f(x)$ (c) (4 points) $f \circ g(2)$ (d) (4 points) $g \circ g(\pi)$

   [Note: Use radian measure for calculating value of $\cos x$.]

   Solution:

   (a) $f \circ g(x) = f(g(x)) = f(\pi/x) = \cos(\pi/x)$.

   (b) $g \circ f(x) = g(f(x)) = g(\cos x) = \pi/\cos x$.

   (c) Using the answer for (a) we get $f \circ g(2) = \cos(\pi/2) = 0$.

   (d) $g \circ g(\pi) = g(g(\pi)) = g(\pi/\pi) = g(1) = \pi/1 = \pi$.

2. The amount in an account after t years $A(t)$ with continuous compounding is given by the following equation : $A(t) = A(0)e^{0.1t}$. Here $A(0)$ is the original amount.

   (a) (4 points) What is the rate of return r ?

   (b) (6 points) How much would be there after 10 years if you start with 10000 dollars ?

   (c) (10 points) Find the doubling time. i.e, how long would it take for, say, 1000 dollars to become 2000? (It doesn't matter how much you start with, though).

   Solution:

   Under continuous compounding the amount grows like Pe^{rt}, where P is the initial amount. Here we see that $r = 0.1$. So rate of return is 10 percent.

   3a) Just plug in $t = 10$. You get $A(10) = 10000e^{0.1(10)} = 10000e^1 = 27182.82$ dollars.

   3b) To get to 2000, we have $2000 = 1000e^{0.1t}$ which gives $2 = e^{0.1t}$

   Taking logarithms of both sides, $ln(2) = 0.1t \implies t = ln2/0.1 = 6.93$ years or about 6 years and 339 days.

3. (a) Find $\lim_{x\to\infty} \frac{x+4}{x^2+5x+4}$.

   (b) Find $\lim_{x\to-3} \frac{x^2+6x+9}{x^2-9}$

   In (a) does the function have a horizontal asymptote? If so, what is it?

   In (b) why can't you just plug in $x = -3$? How is that problem avoided by the limit process?

   Solution:

   3a. The highest power is x^2.

   Dividing above and below by x^2 we get $\frac{(x+4)/x^2}{(x^2+5x+4)/x^2} = \frac{(1/x)+(4/x^2)}{1+(5/x)+(4/x^2)} \to \frac{0}{1}$ as $x \to \infty$.

   So we get $\lim_{x\to\infty} \frac{x+4}{x^2+5x+4} = 0$ and the $x-$ axis with equation $y = 0$ is the horizontal asymptote.

   (You can check that same is true on negative side of $x-$axis also).

(b) NOTE: DIVIDING BY HIGHEST POWER ONLY WHEN $x \to \infty$.

$$\lim_{x \to -3} \frac{x^2 + 6x + 9}{x^2 - 9} = \lim_{x \to -3} \frac{(x+3)^2}{(x+3)(x-3)} = \lim_{x \to -3} \frac{x+3}{x-3} = 0/(-6) = 0.$$

You can't just plug in -3 because you get 0/0 which is undefined.

The limit process avoids this by looking only at points *near* -3.

4. Find the derivative of $f(x) = \sqrt{x}$ at $x = 1$ using limit formula. In other words, find $f'(1)$.

You MUST use limit formula. 6 points for just writing down formula.

Solution:

From the limit formula we have

$$f'(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0} \frac{\sqrt{1+h} - \sqrt{1}}{h} = \lim_{h \to 0} \frac{\sqrt{1+h} - 1}{h}$$

Multiplying and dividing by the conjugate, namely $\sqrt{1+h} + 1$,

$$f'(1) = \lim_{h \to 0} \frac{\sqrt{1+h} - 1}{h} = \lim_{h \to 0} \frac{(\sqrt{1+h} - 1)(\sqrt{1+h} + 1)}{(h)(\sqrt{1+h} + 1)}$$

$$= \lim_{h \to 0} \frac{(\sqrt{1+h})^2 - 1^2)}{(h)(\sqrt{1+h} + 1)} = \lim_{h \to 0} \frac{1 + h - 1}{h(\sqrt{1+h} + 1)} = \lim_{h \to 0} \frac{1}{\sqrt{1+h} + 1}.$$

In the last step we cancelled h and that is okay because, when we are trying to find the limit, h is *near* 0 but not equal to 0. You can't just plug in $h = 0$ because you get 0/0 which is undefined. The limit process avoids this by looking only at points *near* 0.

As $h \to 0$, it is easy to see that $\sqrt{1+h} + 1 \to 2$.

So we get $f'(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0} \frac{\sqrt{1+h} - \sqrt{1}}{h} = 1/2$.

5. Give an example of a function that is continuous but not differentiable at $x = 0$. Justify your answer by showing why it is continuous but not differentiable.

Solution: (This was discussed in class).

The absolute value function $|x|$ is continuous at 0 because the limit as $x \to 0$ is 0 and the value is also 0. In general a function is continuous if and only if the value at $x = a$ is the same as the limit that it approaches as $x \to a$.

Please read 9/13 notes pages 2, 3 and 4 for more details as to when a function is continuous and the ways in which it can be discontinuous. It is not differentiable because slopes of secants from left are approaching -1 while slopes of secants from right are approaching +1 so the limit of their slopes does not exist. Indeed, the tangent is not even defined at 0. The derivative is the limit of the slopes of the secants, so the derivative does not exist and we say the function is not differentiable.

9/21/2017 Fall 2017, Calculus I Test I (Version 2) Sitaraman

Howard University Math Department

Instructions:
PLEASE PROVIDE STEP BY STEP EXPLANATIONS
WRITING ONLY ANSWERS WILL NOT GET FULL CREDIT
Time Limit 50 minutes. Total 100 points. Each 20 points.
Please read the questions carefully before answering

1. Given $f(x) = \cos x$ and $g(x) = \frac{x}{\pi}$ find the following:

   (a) (6 points) $f \circ g(x)$ (b) (6 points) $g \circ f(x)$ (c) (4 points) $f \circ g(2)$ (d) (4 points) $g \circ g(\pi)$

   [Note: Use radian measure for calculating value of $\cos x$.]

   Solution:

   (a) $f \circ g(x) = f(g(x)) = f(\pi/x) = \cos(x/\pi)$.

   (b) $g \circ f(x) = g(f(x)) = g(\cos x) = (\cos x)/\pi$.

   (c) Using the answer for (a) we get $f \circ g(2) = \cos(2/\pi) = 0.804$.

   (d) $g \circ g(\pi) = g(g(\pi)) = g(\pi/\pi) = g(1) = 1/\pi$.

2. The amount in an account after t years $A(t)$ with continuous compounding is given by the following equation : $A(t) = 10000e^{0.01t}$.

   (a) (4 points) What is the amount at the beginning ?

   (b) (6 points) How much would be there after 10 years ?

   (c) (10 points) Find the doubling time. i.e, how long would it take for, say, 1000 dollars to become 2000? (It doesnt matter how much you start with, though).

   Solution:

   Under continuous compounding the amount grows like Pe^{rt}, where P is the initial amount. Here we see that $r = 0.01$. So rate of return is 1 percent.

   For orginal amount put $t = 0$. You get $A(0) = 10000e^0 = 10000$ dollars.

   3a) Just plug in $t = 10$. You get $A(10) = 10000e^{0.01(10)} = 10000e^{0.1} = 11051.71$ dollars.

   3b) To get to 2000, we have $2000 = 1000e^{0.01t}$ which gives $2 = e^{0.01t}$

   Taking logarithms of both sides, $ln(2) = 0.01t \implies t = ln2/0.01 = 69.31$ years or about 69 years and 113 days.

3. (a) Find $\lim_{x\to\infty} \dfrac{x^2+4}{x+4}$.

   (b) Find $\lim_{x\to -3} \dfrac{x^2+6x+9}{x^2-9}$

   In (a) does the function have a horizontal asymptote? If so, what is it?

   In (b) why can't you just plug in $x = -3$? How is that problem avoided by the limit process?

   Solution:

   3a. The lowest power is x.

   Dividing above and below by x we get $\dfrac{(x^2+4)/x}{(x+4)/x} = \dfrac{x+(4/x)}{1+(4/x)} \to \infty$ as $x \to \infty$.

So we get $\lim_{x\to\infty} \frac{x^2+4}{x+4} = \infty$ and there is no horizontal asymptote. (You can check that same is true on negative side of $x-$axis also).

You can also do this problem by dividing by the highest power x^2 but then you will be dealing with a 0 in the denominator which makes things a bit more complicated.

(b) NOTE: DIVIDING BY HIGHEST POWER ONLY WHEN $x \to \infty$.

$$\lim_{x\to -3} \frac{x^2+6x+9}{x^2-9} = \lim_{x\to -3} \frac{(x+3)^2}{(x+3)(x-3)} = \lim_{x\to -3} \frac{x+3}{x-3} = 0/(-6) = 0.$$

You can't just plug in -3 because you get 0/0 which is undefined.

The limit process avoids this by looking only at points *near* -3.

4. Find the derivative of $f(x) = \sqrt{x+2}$ at $x = 1$ using limit formula. In other words, find $f'(1)$.

   You MUST use limit formula. 6 points for just writing down formula.

   Solution:

   From the limit formula we have

$$f'(1) = \lim_{h\to 0} \frac{f(1+h+2) - f(1)}{h} = \lim_{h\to 0} \frac{\sqrt{3+h} - \sqrt{3}}{h} = \lim_{h\to 0} \frac{\sqrt{3+h} - \sqrt{3}}{h}$$

   Multiplying and dividing by the conjugate, namely $\sqrt{1+h}+1$,

$$f'(1) = \lim_{h\to 0} \frac{\sqrt{3+h} - \sqrt{3}}{h} = \lim_{h\to 0} \frac{(\sqrt{3+h} - \sqrt{3})(\sqrt{3+h} + \sqrt{3})}{(h)(\sqrt{3+h} + \sqrt{3})}$$

$$= \lim_{h\to 0} \frac{(\sqrt{3+h})^2 - (\sqrt{3})^2)}{(h)(\sqrt{3+h} + \sqrt{3})} = \lim_{h\to 0} \frac{3+h-3}{h(\sqrt{3+h} + \sqrt{3})} = \lim_{h\to 0} \frac{1}{\sqrt{3+h} + \sqrt{3}}.$$

   In the last step we cancelled h and that is okay because, when we are trying to find the limit, h is *near* 0 but not equal to 0. You can't just plug in $h = 0$ because you get 0/0 which is undefined. The limit process avoids this by looking only at points *near* 0.

   As $h \to 0$, it is easy to see that $\sqrt{3+h} + \sqrt{3} \to 2\sqrt{3}$.

   So we get $f'(1) = \lim_{h\to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h\to 0} \frac{\sqrt{3+h} - \sqrt{3}}{h} = \frac{1}{2\sqrt{3}}$.

5. Give an example of a function that is not continuous at $x = 1$ but does not have vertical asymptote and does not oscillate. Justify your answer by showing why it is not continuous and finding the relevant limits. Not enough to just draw a graph.

   Solution: (Examples were discussed in quiz3 and in class).

   From Quiz 3:

   Let

$$f(x) = \begin{cases} x+1, & x < 1, \\ x^2 - 1, & x \geq 1. \end{cases}$$

   To find $\lim_{x\to 1^+} f(x)$ we need to look at what happens to the right of 1.

   Here the function is defined by $x^2 - 1$ which is a continuous function (because it is a polynomial) and approaches 0 as x approaches 1. [Since it is continous the value is obtaind by just plugging in 1]. So the limit is 0.

To find $\lim_{x \to 1^-} f(x)$ we need to look at what happens to the left of 1.

Here the function is defined by $x + 1$ which is also a continuous function and approaches 2 as x approaches 1. As before, since it is continous the value is obtaind by just plugging in 1. So the limit is 2.

Clearly the function does not have a vertical asymptote at $x = 1$ because it does not go to ∞ as $x \to 1$ from left or right.

Since the two limits are not equal, $\lim_{x \to 1} f(x)$ doesn't exist.

From above, we see that it is not continuous at 1 because the limit of function as x approaches 1 doesn't exist. [Since limit doesn't exist, it is automatically discontinuous and we needn't even look at $f(1)$ although just FYI the value of the function at $x = 1$ *which is defined* by $f(1) = 1^2 - 1 = 0$. In general a function is continuous if and only if the value at $x = a$ is the same as the limit that it approaches as $x \to a$. Please read 9/13 notes pages 2, 3 and 4 for more details as to when a function is continuous and the ways in which it can be discontinuous.

Note that we need to use $x^2 - 1$ at $x = 1$.

9/21/2017 Fall 2017, Calculus I Test I (Version 3) Sitaraman

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Time Limit 50 minutes. Total 100 points. Each 20 points.
Please read the questions carefully before answering

1. Given $f(x) = \sin x$ and $g(x) = \pi x$ find the following:

   (a) (6 points) $f \circ g(x)$ (b) (6 points) $g \circ f(x)$ (c) (4 points) $f \circ g(2)$ (d) (4 points) $f \circ f(\pi)$

   [Note: Use radian measure for calculating value of $\sin x$.]

   Solution:

   (a) $f \circ g(x) = f(g(x)) = f(\pi x) = \sin(\pi x)$.

   (b) $g \circ f(x) = g(f(x)) = g(\sin x) = \pi \sin x$.

   (c) Using the answer for (a) we get $f \circ g(2) = \sin(2\pi) = 0$.

   (d) $f \circ f(\pi) = f(f(\pi)) = f(\sin \pi) = f(0) = \sin(0) = 0$.

2. The amount in an account after t years $A(t)$ with continuous compounding is given by the following equation : $A(t) = Pe^{0.01t}$ where P is original amount.

   (a) (8 points) What is the amount at the beginning, if after 1 year there was 10,100 dollars and 50 cents?

   (b) (4 points) How much would be there after 10 years ?

   (c) (8 points) Find the doubling time. i.e, how long would it take for, say, 1000 dollars to become 2000? (It doesnt matter how much you start with, though).

   Solution:

   Under continuous compounding the amount grows like Pe^{rt}, where P is the initial amount. Here we see that $r = 0.01$. So rate of return is 1 percent.

   For orginal amount put $t = 0$. You get $A(0) = Pe^0 = P$ dollars. So we really just need to solve for P.

   It is given that $A(1) = Pe^{0.01(1)} = 10100.5$ dollars.

   So we get $P = 10100.5/e^{.01} = 10000$ dollars.

   Incidentally this also gives the equation $A(t) = 10000e^{0.01t}$ for the amount after t years.

   3a) Just plug in $t = 10$. You get $A(10) = 10000e^{0.01(10)} = 10000e^{0.1} = 11051.71$ dollars.

   3b) To get to 2000, we have $2000 = 1000e^{0.01t}$ which gives $2 = e^{0.01t}$

   Taking logarithms of both sides, $ln(2) = 0.01t \implies t = ln2/0.01 = 69.31$ years or about 69 years and 113 days.

3. (a) Find $\lim_{x\to\infty} \dfrac{x+4}{x-4}$.

   (b) Find $\lim_{x\to-3} \dfrac{x^2+6x+9}{x^2-9}$

   In (a) does the function have a horizontal asymptote? If so, what is it?

   In (b) why can't you just plug in $x = -3$? How is that problem avoided by the limit process?

   Solution:

3a. The highest (and lowest) power is x.

Dividing above and below by x we get $\dfrac{(x+4)/x}{(x-4)/x} = \dfrac{1+(4/x)}{1-(4/x)} \to 1$ as $x \to \infty$.

So we get $\lim_{x\to\infty} \dfrac{x+4}{x-4} = 1$ and $y = 1$ is the horizontal asymptote. (You can check that same is true on negative side of $x-$axis also).

(b) NOTE: DIVIDING BY HIGHEST POWER ONLY WHEN $x \to \infty$.

$$\lim_{x\to-3} \frac{x^2+6x+9}{x^2-9} = \lim_{x\to-3} \frac{(x+3)^2}{(x+3)(x-3)} = \lim_{x\to-3} \frac{x+3}{x-3} = 0/(-6) = 0.$$

You can't just plug in -3 because you get 0/0 which is undefined.

The limit process avoids this by looking only at points *near* -3.

4. Find the derivative of $f(x) = 2\sqrt{x}$ at $x = 1$ using limit formula. In other words, find $f'(1)$.

   You MUST use limit formula. 6 points for just writing down formula.

   Solution:

   From the limit formula we have

$$f'(1) = \lim_{h\to0} \frac{f(1+h)-f(1)}{h} = \lim_{h\to0} \frac{2\sqrt{1+h}-2\sqrt{1}}{h} = \lim_{h\to0} 2\left(\frac{\sqrt{1+h}-1}{h}\right).$$

   Since 2 is a fixed number and is not affected by what happens to h, we can take it out of the limit formula.

   Multiplying and dividing by the conjugate of $\sqrt{1+h}-1$, namely $\sqrt{1+h}+1$,

$$f'(1) = 2\left(\lim_{h\to0} \frac{\sqrt{1+h}-1}{h}\right) = 2\left(\lim_{h\to0} \frac{(\sqrt{1+h}-1)(\sqrt{1+h}+1)}{(h)(\sqrt{1+h}+1)}\right)$$

$$= 2\left(\lim_{h\to0} \frac{(\sqrt{1+h})^2-1^2)}{(h)(\sqrt{1+h}+1)}\right) = 2\left(\lim_{h\to0} \frac{1+h-1}{h(\sqrt{1+h}+1)}\right) = 2\left(\lim_{h\to0} \frac{1}{\sqrt{1+h}+1}\right).$$

   In the last step we cancelled h and that is okay because, when we are trying to find the limit, h is *near* 0 but not equal to 0. You can't just plug in $h = 0$ because you get 0/0 which is undefined. The limit process avoids this by looking only at points *near* 0.

   As $h \to 0$, it is easy to see that $\sqrt{1+h}+1 \to 2$.

   So we get $f'(1) = 2\left(\lim_{h\to0} \dfrac{f(1+h)-f(1)}{h}\right) = 2\left(\lim_{h\to0} \dfrac{\sqrt{1+h}-\sqrt{1}}{h}\right) = 2(1/2) = 1.$

5. Give an example of a function that is not continuous at $x = 0$ but does not have vertical asymptote and does not have limits from either side. Justify your answer by showing why it is not continuous and saying why the relevant limits do not exist. Not enough to just draw a graph.

   Solution: As discussed in class, the function $sin(\pi/x)$ oscillates between -1 and 1 no matter how close you go to 0. So it does not have limits from either side. Since the iimit of the function does not exist at $x = 0$, the function cannot be continuous there.

   In general a function is continuous if and only if the value at $x = a$ is the same as the limit that it approaches as $x \to a$. Please read 9/13 notes pages 2, 3 and 4 for more details as to when a function is continuous and the ways in which it can be discontinuous.