11/3/2017 Fall 2017, Calculus I Quiz 8 Solutions Sitaraman

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1. (10 points) Show that $\lim_{x\to 0} \frac{1-\cos x}{\sin(2x)} = 0$ using L'Hospital's rule.

   First show that the rule is applicable here.

   Solution:

   First we check that L'Hospital's rule applies. We have both $1-\cos x$ and $\sin(2x)$ going to 0 as x goes to 0. Also both are differentiable around 0 and derivative of denominator is not zero near 0. So it applies.

   We get upon applying the rule,

   $\lim_{x\to 0} \frac{1-\cos x}{\sin(2x)} = \lim_{x\to 0} \frac{(1-\cos x)'}{(\sin(2x))'} = \lim_{x\to 0} \sin x/(2\cos(2x)) = 0/2 = 0.$

2. (20 points) A cardboard box with a square base and open top must have exactly 108 cubic meters volume. Show that when length of base is 6 meters the area is minimized (hence also the amount of cardboard). What is the height when length is 6 ?

   Need to set up problem with x as length of base and then minimize $A(x)$ and show that $x = 6$ when $A(x)$ is minimum.

   The first thing to do in this problem is to figure out what it is that we are minimizing. Usually it is obvious from the question itself. Here it says the material must be maximized. That is same as minimizing the surface area. Call it $A(x)$ where x can be the length (as well as width, because it is square) of the base. Let y be the height of the box. This is just our choice – you can choose the height to be the main variable as well, although sometimes one may be easier to work with than the other.

   The next thing to do is to write $A(x) = x^2 + 4xy$. Proof: Total area is Area of base plus four times area of any side because there is no top. We need to write this in terms of the one variable x.

   Now we have volume is fixed at 108: $x^2y = 108$. (product of length, width and height)

   Solving for y we get $y = 108/x^2$.

   Plugging this into the formula for A we can write A as a function of x :

   $A(x) = x^2 + 4xy = x^2 + 4x(108/x^2) = x^2 + (432/x)$.

   Next we need to find the critical points.

   We set $A'(x) = 0$ and solve for x :

$$A'(x) = \left(x^2 + \frac{432}{x}\right)' = 0 \implies 2x = -432\left(\frac{-1}{x^2}\right) \implies 2x^3 = 432 \implies x = 6.$$

Then plug in this value of x into the formula for y. Get $y = 108/6^2 = 3$. So the height is 3 meters.

Now the second derivative is $A''(x) = (2x - (432/x^2))' = (2 + (864/x^3))$. This is clearly positive when $x = 6$, so the area and hence material used is minimum at this value.

You can see that the Area function is always concave up when x is positive. Also the area at the boundary points $x = 0$ and $x = \infty$ goes to ∞ in both cases (why are these the boundary points?) so the area is absolute minimum at $x = 6$.

3. (10 points extra credit) Graph the area function for the box in problem 2 showing clearly its intercepts, maxima and minima and asymptotes if any. You only need to graph it on the positive side of $x-$axis.

Solution:

There is no y intercept for $A(x) = x^2 + (432/x)$ because at $x = 0$ function is undefined.

In fact the $y-$axis is a vertical asymptote.

The $x-$intercepts are when $x^2 + (432/x) = 0 \implies (x^3 + 432)/x = 0 \implies x^3 = -432$.

This has no solutions on the positive side of $x - axis$.

The derivative is $A'(x) = 2x - (432/x^2)$ and it is negative when $x < 6$ and positive afterwards.

$A''(x) = (2 + (864/x^3))$ is always positive so it is always concave up.

Graph is below.