11/30/16 Fall 2016, Intro to Modern Algebra I Test III Solutions Sitaraman

EACH PROBLEM 20 POINTS. ANSWER AS MANY AS YOU CAN.

1. If G_1 and G_2 are two groups show that $G_1 \times G_2 \simeq G_2 \times G_1$.

Solution: This was a homework problem. The map is given by taking (a,b) to (b,a).

2. Show that $\mathbb{R}^*$, the non-zero real numbers under multiplication is the inner direct product of the subgroup $H = \{1, -1\}$ and $K = \mathbb{R}^{>0}$ the subgroup of positive real numbers.

Solution: It is easy to see that $H \cap K = \{1\}$ and that $\mathbb{R}^* = HK$. Since $\mathbb{R}^*$ is abelian both subgroups are normal. So by corollary to Theorem 2.9.4, $\mathbb{R}^*$ is the inner direct product of H and K.

3. Give one example of a normal subgroup and one example of a subgroup that is not normal for the group S_3.

Solution: As shown in class, the subgroup generated by the cycle (123) is normal and any of the three subgroups generated by the transpositions (12), (13) or (23) are not normal.

4. Write (12345)(3567) and (145)(256) as a product of disjoint cycles.

(12345)(3567) = (123)(4567) and (145)(256) = (14562).

5. Find the orders of (1456)(32) and (14356)(279).

Solution: (1456)(32) has order 4 because we have a product of a four cycle and a two cycle that are disjoint and they are of order 4 and 2 respectively. Taking LCM of 4 and 2 we get the answer. You can also multiply and find order. $((1456)(32))^2 = (15)(46)(32)$. This is a product of disjoint 2-cycles and its square will be identity. So order is 2 times 2 = 4.

(14356)(279) has order 15 because we have a product of a five cycle and a three cycle that are disjoint.

6. What are the possible orders of permutations in S_4 ? [Hint: What kind of cycle decompositions are possible?].

Solution:

1. With Lagrange's theorem: The order of an element divides the order of the group. S_4 has 24 elements, so the possible orders are 1, 2, 3, 4, 6, 12 and 24.

Of these, 12 and 24 are not possible because at the most we have a four cycle. You cannot have a product of a four cycle and a 3 cycle that are disjoint because there are only a total of four numbers.

Even six is not possible because the only way that can happen is if you have a product of a three cycle and a two cycle that are disjoint.

So only 1,2,3 and 4 are possible and it is easy to write down the kind of permutations that have those orders.

2. Without Lagrange's theorem: As we did above, just look at the possible ways a permutation on 1,2,3 and 4 can be broken into disjoint cycles. You can either have a 2-cycle, a 3-cycle, a 4-cycle, or a product of two 2-cycles.

FYI, there are six 2-cycles, six 4-cycles, eight 3-cycles, three that are products of two 2-cycles and then the identity permutation. Prove this!