Fall 16 Intro to Modern Algebra I Final exam solutions Sitaraman 12/9/16

SHOW ALL WORK. EACH 20 POINTS UNLESS OTHERWISE SPECIFIED.

1. Prove that the set of integers with the operation $m \circledast n = m + n - mn$ is not a group. Which axioms are satisfied and which are not?

Solution: Existence of inverse is not satisfied for some integers under this operation.

2. What are the possible orders of the subgroups of S_4? Find a subgroup of order 6 in S_4.

Solution: By Lagrange's theorem they are all the numbers that divide $24 = 4!$, the order of S_4. They are 1,2,3,4,6,8,12,24.

The group S_3 is actually a subgroup of S_4 if you let every permutation in S_4 act on 1,2,3,4 by fixing 4 and permuting 1,2,3 as it does in S_3.

3. Show that a subgroup of index 2 in any group is a normal subgroup of that group. Show that A_4, the subgroup of even permutations that is of order 12 in S_4, is of index 2 and hence normal. Show also that A_4 is not abelian.

Solution: The first part was done in class. It is of index 2 because $|S_4|/|A_4| = 24/12$. (123) and (12)(34) are both even permutations, so they are in A_4. (123)(12)(34) = (134) but (12)(34)(123) = (243).

4. Show that any group of prime order must be cyclic.

Solution: Since the order of an element must divide order of the group, the order of any non-identity element has to equal the prime number that is the order of the group. So the group is cyclic, generated by any non-identity element.

5. Give an example of a non-trivial cyclic subgroup and a nontrivial subgroup that is not cyclic in S_4, both of order 4.

Solution: Any subgroup generated by a single permutation is a cyclic subgroup. The subgroup generated by (12) and (34) is a non-cyclic subgroup of order 4.

6. Give an example of a non-trivial normal subgroup of D_8 (the dihedral group of order 8). D_8 is generated by the permutations (1234) and (13).

Solution: The subgroup generated by (1234) will be of order 4 and hence of index 2, so it would be normal. BTW the elements in D_8 are (1234), $(1234)^2 = (13)(24), (1234)^3 = (1432)$, $(1234)(13) = (14)(23), (13)(1234) = (12)(34), (1234)^2(13) = (13)(24)(13) = (24)$.

7. Show that $\mathbb{Z}_{12}$ cannot be isomorphic to the direct product $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$.

Solution: If they are isomorphic, that means $\mathbb{Z}_{12}$ will have a subgroup isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$ because, as shown in class, if you have a direct product of two groups $G_1 \times G_2$ then the direct product has a subgroup isomorphic to each group given by elements of the form $(g_1, 0)$ and $(0, g_2)$. But $\mathbb{Z}_{12}$ cannot have a subgroup isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$ because that means it will have two elements of order 2 but it has only one element of order 2, namely 6.

Alternatively, $\mathbb{Z}_{12}$ is cyclic and every subgroup of a cyclic group is cyclic whereas $\mathbb{Z}_2 \times \mathbb{Z}_2$ is not cyclic.

8. Describe the cosets of the subgroup $H = \{1, 3, 9\}$ in $G = U_{13} = \mathbb{Z}_{13}^*$.

Solution: The cosets will be $H, 2H, 4H, 8H$.

9. Using an appropriate homomorphism, show that $G/H \simeq \mathbb{Z}_4$ where G and H are as in problem 8.

Solution: From the description of the cosets you see that it is generated by $2H$. Mapping $2H$ to 1 (the generator of $\mathbb{Z}_4$) we get the isomorphism. So $4H \to 2, 8H \to 3, H \to 0$.

10. Show that kernel of any homomorphism is a normal subgroup. Give another proof that A_4 is normal by showing that it is the kernel of a homomorphism to the group {1,-1}.

Solution: The homomorphism is given by mapping even permutations to 1 and odd ones to -1.

11. State Cauchy's theorem. Use it to show that $\mathbb{Z}_{12}$ will have an element of order 6.

Solution: $\mathbb{Z}_{12}$ has order 12, divisible by the primes 2 and 3. So it has elements of order 2 and 3. SInce it is abelian they commute and so the order of their product is 6 (lcm of 2 and 3).

12. Let G be any group and $a, b \in G$. Show that the relation $a \sim b$ if $a = xbx^{-1}$ for some $x \in G$ is an equivalence relation. The equivalence classes are the conjugacy classes. Decompose S_3 into disjoint conjugacy classes.

Solution: The conjugacy classes are $\{identity\}, \{(12), (23), (13)\}, \{(123), (132)\}$.