10/31/07 Fall 07, Intro to Modern Algebra I Test II solutions Sitaraman

1.Is the set of rational numbers with odd number denominators a subgroup of the set of all rational numbers under addition? Is it a subgroup under multiplication? (in this case we consider only non-zero rational numbers)

soln: Yes, because the sum of two rational numbers with odd denominators will also have an odd denominator (closure) and the negative of such a fraction will also have odd denominator (existence of inverse). Under multiplication it is closed but the inverse (in this case, reciprocal) of a fraction with odd denominator might not have odd denominator.

2. Prove if true or give counter-example if false: (a) The union of two subgroups of a group is also a subgroup (b) Every subgroup of an abelian group is abelian

soln: (a) is false. In the group of integers if you take the union of the even integers and the multiples of 3 the result is not a subgroup. For instance, 2 is even and 3 is a multiple of 3 but 2+3 = 5 is not in either set. (b) True. Proof is easy. In fact, it is true that any subset of an abelian group is abelian. Because $ab = ba$ is true for any two elements a, b in an abelian group.

3. List all the subgroups of U(20). Which of them are cyclic? If they are cyclic, produce a generator.

U(20) consists of 1,3,7,9,11,13, 17, and 19. Its subgroups are $< 3 >$ the cyclic subgroup generated by 3, of order 4, containing 1,3,9 and 7; The cyclic subgroup generated by 13, also of order 4, containing 13, 17, 9, and 1. The cyclic subgroups generated by 9, 11 and 19 all of which have order 2 and thus those subgroups have order 2 as well. There is also a subgroup consisting of 1,9,11,19 which is not cyclic (because every element is of order 2). U(20) itself is a subgroup of U(20) but it is not cyclic because there is no element of order 8. *Note that some subgroups can be cyclic even though the group itself is not.* Finally, we have the subgroup consisting of just the identity element, namely 1.

4. List all the elements of order 10 in $\mathbf{Z}_{240}$. List the elements of the unique cyclic subgroup of order 10 in the same group.

Soln: For this problem we use theorem 4.2,4.3 and 4.4 and their corollaries. $\mathbf{Z}_{240}$ under addition is cyclic, generated by 1. It has a unique subgroup of order 10, namely the cyclic subgroup generated by 24 given by $< 24 >= \{24, 48, 72, 96, 120, 144, 168, 192, 216\}$. Note that not all the elements in this subgroup are of order 10. Thus not all of them are generators of this subgroup. There are exactly $\phi(10) = 4$ elements of order 10, namely $24, 24^3 = 3(24) = 72, 24^7 = 4(24) = 168, 24^9 = 9(24) = 216$. They are also generators because they are of order 10. Notice that we took the powers that are relatively prime to 10, namely 1,3,7 and 9. Also here the powers are just the multiples because the operation is addition.

5. List all the elements (in cycle notation) in the cyclic subgroup generated by (1234) in S_4. What is the order of this subgroup? What are the orders of the elements of this subgroup?

soln: (1234) is of order 4 because it is a 4 cycle. Its subgroup contains $(1234), (1234)^2 = (13)(24), (1234)^3 = (1432)$ and the identity permutation. (13)(24) has order 2 because it is a product of disjoint 2 cycles. All the rest have order 4 because they are 4 cycles. The inverse of (1234) is $(1234)^3 = (1432)$.

6. Write down the following permutation as (i) a product of disjoint cycles (ii) a product of 2 cycles. Find its order and its inverse:

$$\begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 7 & 5 & 4 & 1 & 2 & 3 & 6 \end{matrix}$$

Soln: Given permutation is (17634)(25). (17634) is a 5 cycle, therefore of order 5. using 2 cycles, it can be written as (14)(13)(16)(17)(25). The order of the permutation is lcm(2,5) = 10. Its inverse is

$(17634)^{-1}(25)^{-1} = (17634)^4(25) = (13746)(25)$.