10/1/07 Fall 02, Intro to Modern Algebra I Test I Sitaraman

Instructions: Each problem carries 15 points. Do as many as you can, within 50 minutes. Anything you get above 100 is extra credit. You must provide step by step explanations to get partial credit.

1. Prove by induction: The sum of the first n odd numbers for any n given by $1 + 3 + \ldots + (2n - 1)$ equals n^2.

2. Show that the relation $a \sim b$ if $a+b$ is even where a, b are integers is an equivalence relation on the set of integers, and describe the equivalence classes. What is the equivalence class of 0? How many equivalence classes are there?

3. Find $12345678^6 \bmod 9$ by using the divisibility test for 9 (same as used in the checksum procedure) and the fact that products of remainders are the same as remainders of products. (in this case, that helps you to find the remainder of the power by taking the power of the remainder).

4. Describe $U(20)$ and write the Cayley table for it. Read off the inverse of each element from the Cayley table.

5. Prove that in any dihedral group D_n with $n \geq 3$, the rotation taking each vertex to its adjacent vertex (i.e, 1 to 2, 2 to 3, etc.,) does not commute with reflection through any of the diagonals. This shows that D_n is never commutative (or abelian).

6. Give an example of each of the following:

(a) An infinite group that is not commutative (Hint: matrices). (b) A finite group with more than 3 elements that is commutative (c) An element of finite order in an infinite group.

7. Show that the set of 2×2 matrices whose entries are from $\mathbf{Z}_5$ and the operation is matrix addition (modulo 5, of course) is a finite, abelian group. How many elements does it have?

8. For any element g in any group G, show that the inverse of g^k is the same as $(g^{-1})^k$, where k is a positive integer.