12/3/2015 Fall 2015, Calculus I Test III Sitaraman

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1. (20 points) Graph the function $y = \frac{1}{x^2+1}$. You must explain how you got the graph.

   You must find the following:

   a) All the local maxima and minima and inflexion points.

   b) Where it is increasing, decreasing, concave up and concave down.

   Solution:

   We have, using chain rule, $y' = \dfrac{-1}{(x^2+1)^2}(2x)$. This is zero when numerator $-2x = 0$ which hapens only at 0. It is defined everywhere because denominator is never zero because x^2+1 has no real zeroes. So 0 is the only critical point.

   The second derivative is obtained using quotient rule and after simplifying it becomes $\dfrac{6x^4+4x^2-2}{(x^2+1)^4}$.

   This is zero when the numerator is zero. The numerator is a quadratic function $6u^2+4u-2=0$ if you set $u = x^2$. Dividing by 2 it becomes $3u^2+u-1=0 \implies (3u-1)(u+1)=0$.

   $u+1 = x^2+1$ gets cancelled by the denominator.

   So the only zeroes are when $3u-1 = 3x^2-1 = 0 \implies x = \pm 1/\sqrt{3}$.

   [The second derivative is a bit easier to find here if you write the first derivative as $(-2x)(x^2+1)^{-2}$ and use product rule to differentiate it].

   These are the two inflexion points.

   As x goes to plus or minus infinity the denominator goes to infinity and so the function goes to 0. So the graph is asymptotic to $x-$axis on both ends.

   At the critical point $x = 0$ the second derivative is -2 so it is a local maximum.

   Also the derivative is clearly negative on the negative side of the $x-$axis and positive on the positive side.

   The second derivative is negative in the interval between the two inflexion points and positive outside it.

   So it starts at 0 from negative infinity and slowly increases to 1 at $x = 0$. In between it changes from concave up to concave down at $-1/\sqrt{3}$. Then it changes from concave down to concave up again at $1/\sqrt{3}$. It decreases from 0 to infinity and goes to zero as x approaches positive infinity. We also see that $x = 0$ is the absolute maximum and the funciton equals 1 there.

2. (15 points) Show that $\lim\limits_{x\to 0} \dfrac{\sin x^2}{x} = 0$ using L'Hospital's rule.

   Solution:

   First we check that L'Hospital's rule applies. We have both $\sin x^2$ and x going to 0 as x goes to 0. Both functions are differentiable and derivative of denominator is not zero near 0. So it applies.

   We get upon applying the rule, $\lim\limits_{x\to 0} \dfrac{\sin x^2}{x} = \lim\limits_{x\to 0} \dfrac{(\sin x^2)'}{(x)'} = \lim\limits_{x\to 0} \dfrac{(\cos x^2)(2x)}{1} = \cos(0)(0) = 1 \times 0 = 0.$

3. (20 points) Find the dimensions of the rectangle that has the largest area among all the ones with perimeter $2L$. Your answer will be in terms of the variable number L.

   First we need to find the function that is to be maximized.

   Since it says "largest area" we maximize $A(x) = xy$ where x is the length and y is width.

   [It will work if you choose x to be the width, as well].

We need to make $A(x)$ really a function of just x alone, and to do that we need to solve for y in terms of x.

Perimeter is constrained to be equal to $2L$, so we get $2x + 2y = 2L \implies y = L - x$.

Plugging this in, we get $A(x) = xy = x(L - x) = xL - x^2$.

This is defined everwhere. It is zero when $xL - x^2 = x(L - x) = 0 \implies x = 0$ or $x = L$.

So the boundary points are 0 and L because otherwise $A(x)$ will be negative and area cannot be negative.

Critical points: $A'(x) = 0 \implies L - 2x = 0 \implies x = L/2$.

Since the function is defined everywhere $L/2$ is the only critical point.

Checking for local max / min:

First try second derivative test.

$A''(x) = -2$.

This is negative always, so $L/2$ is a local maximum.

At $x = L/2, y = L - x = L - (L/2) = L/2$ and $A(x) = xy = (L/2)(L/2) = L^2/4$.

Comparing the value at the critical point with the values at the boundaries 0 and L (both of which are 0) we get that the square with length and width equal to $L/2$ will have maximum area among all rectangles of perimeter $2L$.

4. (10 points extra credit) Approximate the area under $f(x) = x^2$ from $x = 0$ to $x = 1$ by using a Riemann sum on the left endpoints with 5 intervals of equal width. Compare it with actual value found using fundamental theorem of calculus (i.e, using antiderivative).

   Solution:

   The desired area is given by $\int_0^1 x^2 \; dx$

   For the Riemann sum $n = 5$ and $\Delta x = (1 - 0)/5 = 0.2$.

   The integral is approximated by

$$\sum_{k=0}^{4} (1 + k(0.2)^2(0.2) = (0.2)(0^2 + 0.2^2 + 0.4^2 + 0.6^2 + 0.8^2) = (0.2)(1.2) = 0.24.$$

   Actual value is $\int_0^1 x^2 dx = \left[\frac{x^3}{3}\right]_0^1 = (1/3)(1^2 - 0^2) = 0.33$

   (The approximation will get better as we increase the number of intervals).

5. (15 points) Find the area under $y = \frac{x^2 + \sqrt{x} + 1}{x}$ from $x = 1$ to $x = 4$.

   Solution: The area is given by:

$$\int_1^4 \frac{x^2 + \sqrt{x} + 1}{x} \; dx = \int_1^4 \frac{x^2}{x} \; dx + \int_1^4 \frac{\sqrt{x}}{x} \; dx + \int_1^4 \frac{1}{x} \; dx =$$

$$\int_1^4 x \; dx + \int_1^4 x^{-1/2} \; dx + \int_1^4 \frac{1}{x} \; dx = \left[x^2/2 + \frac{x^{-(1/2)+1}}{(-1/2) + 1} + lnx\right]_1^4$$

$$== \left[x^2/2 + \frac{x^{1/2}}{1/2} + lnx\right]_1^4 = \left[4^2/2 + 2(4^{1/2}) + ln4\right] - \left[1^2/2 + 2(1^{1/2}) + ln1\right]$$

$$= (8 + 4 + ln4) - (1/2 + 2 + 0) = 9.5 + ln4 = 9.5 + 1.386 = 10.886 \text{ approximately}$$

6. (20 points) Integrate by substitution: $y = \int \frac{(lnt)^2}{t}\ dt$.

   Solution:

   Let $u = lnt$. Then $du = \frac{dt}{t}$.

   Now we get the integral in terms of u by plugging in $u = lnt$ and $du = \frac{dt}{t}$.

   We get $y = \int u^2\ du = \frac{u^3}{3} + C = \frac{(lnt)^3}{3} + C$.

7. (10 points) Find the derivatives of the functions defined by

   (a) $f(x) = \int_0^x cos(e^t)\ dt$. (b) $f(x) = \int_x^1 \tan^{-1}(t^2)\ dt$.

   Solution:

   (a) Using fundamental theorem of calculus, the derivative is given by $f'(x) = \cos e^x$.

   (b) First write the integeral as $f(x) = -\int_1^x \tan^{-1}(t^2)\ dt$. Now using fundamental theorem we get $f'(x) = -\tan^{-1}(x^2)$.