Howard University Math Department

Instructions:
PLEASE PROVIDE STEP BY STEP EXPLANATIONS
WRITING ONLY ANSWERS WILL NOT GET FULL CREDIT
Time Limit 100 minutes
Please read the questions carefully before answering
Challenge problem is extra credit 20 points.
Total 100 points.

1. (40 points]) Find the derivatives of the following functions:

   $(a)\ x^3e^x \qquad (b)\ sin(3x^2+1) \qquad (c)\ \dfrac{x+e^x}{x} \qquad (d)\ cos^{-1}x + ln(x^2+1)$

   Solution:

   (a) Use product rule to get $f'(x) = (x^3)'e^x + (x^3)(e^x)' = 3x^2e^x + x^3e^x$.

   (b) Use chain rule with $y = \sin(u),\ u = 3x^2+1$ and $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx} = \cos(u)(6x) = 6x\cos(3x^2+1)$.

   (c) Use quotient rule: $\left(\dfrac{x+e^x}{x}\right)' = \dfrac{(x+e^x)'x - (x+e^x)(x)'}{x^2} = \dfrac{(1+e^x)x - (x+e^x)(1)}{x^2} = \dfrac{e^x(x-1)}{x^2}$

   (d) $f'(x) = (cos^{-1}x)' + (ln(x^2+1))' = \dfrac{-1}{\sqrt{1-x^2}} + \dfrac{2x}{x^2+1}$.

2. (20 points) Use implicit differentiation to find dy/dx given that $xy^2 + y^2 - x = 0$.

   Solution:

   Differentiating both sides with respect to x we get

   $$y^2 + x(2yy') + 2yy' - 1 = 0 \implies (2xy+2y)y' = 1 - y^2 \implies y' = \frac{1-y^2}{2xy+2y}.$$

3. (20 points) The population of a town after t years is given by the formula $P(t) = 3t + 5t^{3/2} + 6000$. Using differentials, estimate the change in the population during a period of 3 months (0.25 years) at the end of 5 years. (i.e from $t = 5$ to $t = 5.25$.)

   Solution: The estimate is given by the formula $dP = P'(5)dt = P'(5)(0.25)$.

   We have $P'(t) = 3 + 5(\frac{3}{2}t^{\frac{3}{2}-1})$.

   So $dP = P'(5)(0.25) = \left(3 + \dfrac{15}{2}5^{\frac{1}{2}}\right)(0.25) = 4.94$

4. (20 points) Find the equation of the tangent line to the graph of $f(x)$ where $f(x) = y = \frac{x-1}{x+1}$ at a general point $x = a$. Find the points where it has a horizontal tangent line or the slope of the tangent is undefined.

   Soln: Slope of the tangent at $x = a$ is $f'(a)$. We find $f'(x)$ using quotient rule: $f'(x) = \frac{(x-1)'(x+1)-(x-1)(x+1)'}{(x+1)^2} = \frac{2}{(x+1)^2}$. So the slope at $x = a$ is $2/(a+1)^2$ and the equation is $y - f(a) = \frac{2}{(a+1)^2}(x-a)$ where $f(a) = (a-1)/(a+1)$. The slope is never zero because numerator is 2 and hence never equal to 0. It is undefined when $a = -1$.

5. Find the derivative of $f(x) = (x^2+x+1)(x^3-2x)$ using product rule.

   Soln: $f'(x) = (x^2+x+1)'(x^3-2x) + (x^2+x+1)(x^3-2x)' = (2x+1)(x^3-2x) + (x^2+x+1)(3x^2-2) = 2x^4 - 4x^2 + x^3 - 2x + 3x^4 - 2x^2 + 3x^3 - 2x + 3x^2 - 2 = 5x^4 + 4x^3 - 3x^2 - 4x - 2$.

6. Given that $f(1) = 1$, $f'(1) = -1$, $g(1) = 2$, and $g'(1) = 1$, find the derivative of $f(x)g(x)$ at $x = 1$.

   Soln: $(fg)' = f'g + g'f$. So $fg'(1) = f'(1)g(1) + g'(1)f(1) = (-1)(2) + (1)(1) = -1$.

7. Find the derivative of $sin(x^2)$ and $(x^2 + 1)^{10}$ using chain rule.

   Soln: $(sin(x^2))' = cos(x^2)(x^2)' = 2xcos(x^2)$.
   $((x^2 + 1)^{10})' = 10(x^2 + 1)^9(x^2 + 1)' = 10(x^2 + 1)^9(2x) = 20x(x^2 + 1)^9$.

8. Using implicit differentiation [WITH RESPECT TO y] find $\frac{dx}{dy}$ given that $cosx = y$. Write your answer so that $\frac{dx}{dy}$ is in terms of y. Using this find the derivative of $cos^{-1}y$.

   Soln:Differentiating both sides of $cosx = y$ with respect to y we get $-sinx\frac{dx}{dy} = 1$. Solving for $\frac{dx}{dy}$ we get $\frac{dx}{dy} = -1/sinx$. Since $cosx = y$, we get $sinx = \sqrt{1-y^2}$ and thus $\frac{dx}{dy} = -1/\sqrt{1-y^2}$. Now from $cosx = y$ we get $x = cos^{-1}y$ and $\frac{dx}{dy} = -1/\sqrt{1-y^2}$. So the derivative of $cos^{-1}y$ is $-1/\sqrt{1-y^2}$.

9. Differentiate using logarithms: $\frac{x^{sinx}}{(x^2+1)^{1/3}}$.

   Soln: Let $y = \frac{x^{sinx}}{(x^2+1)^{1/3}}$. Then $lny = ln[x^{sinx}/(x^2 + 1)^{1/3}]$. Using properties of logarithms, we get $lny = ln(x^{sinx}) - ln(x^2+1)^{1/3} = sinxlnx - (1/3)ln(x^2+1)$. Differentiating both sides with respect to x we get $\frac{1}{y}\frac{dy}{dx} = cosxlnx + sinx(1/x) - (1/3)\frac{2x}{x^2+1}$. Solving for $\frac{dy}{dx}$ we get $\frac{dy}{dx} = y[cosxlnx + \frac{sinx}{x} - (1/3)\frac{2x}{x^2+1}]$. Plug in $y = x^{sinx}/(x^2 + 1)^{1/3}$ to get the answer.

10. A rocket is at a height of $h(t)$ at time t seconds. An observer from a distance of 1000 meters measures the angle of elevation as θ at time t. Find the velocity of the rocket after 10 seconds if the angle of elevation is $\pi/4$ radians and changing at 0.1 radian per second.

    Soln: (a) We have $h = 1000tan\theta$. Differentiating both sides with respect to t we get $\frac{dh}{dt} = 1000sec^2\theta\frac{d\theta}{dt} = 1000(sec^2(\pi/4))(0.1) = 200$ meters per second.

11. Find the critical points of $x^{1/3}(x+1)$. Explain whether they are relative maxima or minima using first or second derivative tests.

    Soln: $f'(x) = (x^{1/3})'(x + 1) + (x^{1/3})(x + 1)' = (1/3)(x^{-2/3})(x + 1) + (x^{1/3})(1)$. Simplifying with a common denominator we get $f'(x) = \frac{x+1+3x}{3x^{2/3}} = \frac{4x+1}{3x^{2/3}}$. This is undefined at $x = 0$ and equals 0 at $x = -1/4$. Now we try to classify these critical points as relative maxima or minima using 1st derivative test. Note that the denominator of $f'(x)$ is always positive because it is $(x^{1/3})^2$. The numerator is negative to the left of -1/4 annd positive to the right of -1/4. So $f(x)$ is decreasing upto -1/4 and always increasing after -1/4 (except at 0 where derivative is undefined). So -1/4 is a relative minimum and 0 is neither a relative minimum nor maximum.

12. An object is moving such that its position at time t is given by $s(t)$. State whether the following are true or false. If true explain why and if false disprove it or provide counterexample: (a) The speed is given by $s'(t)$. (b) The acceleration is given by $s''(t)$. (c) The velocity is always positive. (d) The distance traveled from time t_1 to t_2 is given by $s(t_2) - s(t_1)$.

    Soln: (a) False. Speed is the absolute value of velocity which is $s'(t)$. (b) True. Acceleration is rate of change of velocity, i.e, $(s'(t))' = s''(t)$. (c)False. Speed is always positive. (see (a)). (d) False. The distance is given by the absolute value of the same: $|s(t_2) - s(t_1)|$. By itself $s(t_2) - s(t_1)$ simply gives change in position and it could be positive or negative whereas distance is always positive.

13. The population of a town after t years is given by $P(t) = \dfrac{1200}{1+e^{-0.1t}}$. Find the rate of change of this population after 5 years. Is it always increasing or decreasing? What happens to the population as t approaches ∞ ?

Solution: Rate of change after 5 years is given by $P'(5)$. We have, using chain rule:

$$P'(t) = \left(\frac{1200}{1+e^{-0.1t}}\right)' = 1200\left((1+e^{-0.1t})^{-1}\right)' = 1200(-1)(1+e^{-0.1t})^{-2}(-0.1)e^{-0.1t} = 120\frac{e^{-0.1t}}{(1+e^{-0.1t})^2}$$

So $P'(5) = 120e^{-0.5}(1+e^{-0.5})^{-2} = 28.2$.

You can see from the formula for $P'(t)$ that the derivative is always positive. This is because the exponential function is always positive. So the function is always increasing. As $t \to \infty$ we have $e^{-0.1t} = 1/e^{0.1t} \to 0$. So $P(t) \to 1200/(1+0) = 1200$. This is called the carrying capacity of the town or habitat where the populaion is measured. This equation is based on the logistic model of population growth.

14. Find the differential of $f(x) = x^{\frac{1}{3}}$. Use your result to estimate $126^{\frac{1}{3}}$.

Solution:

We have $df = f'(x)dx = \frac{1}{3}x^{\frac{1}{3}-1}dx = \frac{1}{3}x^{-2/3}dx$.

To estimate $126^{1/3}$ note that $125^{1/3} = 5$. When $x = 125, dx = 1$, we get $df = \frac{1}{3}x^{-2/3}dx = \frac{1}{3}(125^{1/3})^{-2} = 1/75 = 0.013333$.

Now use the formula $f(x) \simeq f(x_0) + df$ to get $126^{1/3} \simeq 5 + 0.013333 = 5.013333$.

Compare with actual answer: $126^{1/3} = 5.013298$. smallskip

15. Two cars leave from the same intersection. A goes north and B goes east. A is moving at 10 mph and car B is moving at 20 mph after 3 hours.. Find the rate of change of the distance between them, at that time, given that A and B are both 100 miles away from the intersection. (This is the same as the speed of once car relative to the other at that instant).

Imagine the northbound car A on the positive $y-$axis and the eastbound car B along the positive $x-$axis. A is at 100 miles and B is at 100 miles from the intersection after 3 hours. Using Pythagoras theorem the distance s between them is given by $s^2 = a^2 + b^2$ where A is at a distance a and B is at a distance b from the intersection. Differentiating both sides with respect to t we get $2s(ds/dt) = 2a(da/dt) + 2b(db/dt) \implies ds/dt = (a(da/dt) + b(db/dt))/s$.

So at 3 hrs $a = b = 100$. We also have that $da/dt = 10$ and $db/dt = 20$ at that time. Moreover $s = \sqrt{a^2+b^2} = \sqrt{100^2+100^2} = \sqrt{2}(100) = 141.42$ miles approximately.

So $ds/dt = ((100)(10) + (100)(20))/(141.42) = 3000/141.42 = 21.21$ mph approximately.

16. Find the critical points of the function $y = xe^{-x^2}$.

Soln:$y' = x'e^{-x^2}) + x(e^{-x^2}(-2x)) = e^{-x^2}(1-2x^2)$. We have $y' = 0$ when $1 - 2x^2 = 0$ and so $x = \frac{1}{\sqrt{2}}$ and $x = \frac{-1}{\sqrt{2}}$. The derivative $y' = e^{-x^2}(1-2x^2)$ is well defined everywhere, so there are no critical points because of vertical or undefined tangent lines. So $x = \frac{1}{\sqrt{2}}$ and $x = \frac{-1}{\sqrt{2}}$ are the only critical points.

17. (20 points) Show that the absolute maximum is 123 and absolute minimum is -37 for

$f(x) = -2x^3 + 3x^2 + 12x - 5$ in the interval $-4 \leq x \leq 4$. You must find all critical points.

Solution:

The critical points are found by looking at x where $f'(x)$ is zero or undefined.

We have $f'(x) = -6x^2 + 6x + 12$. This is well defined everywhere.

Setting $f'(x) = 0$ we get $-6x^2 + 6x + 12 = 0 \implies x^2 - x - 2 = 0 \implies (x-2)(x+1) = 0$. Thus we get $x = 2$ and $x = -1$ as the critical points.

Comparing the values of $f(x)$ at the critical points and the boundary points, we get

$f(-4) = 123, f(-1) = -12,\ f(2) = 15, f(4) = -37$. Thus absolute maximum is 123, attained at $x = -4$.

Absolute minimum is -37, attained at $x = 4$.

Additional notes:

You can see that $f'(x)$ is negative and $f(x)$ is decreasing between -4 and -1.

Similarly, between -1 and 2 it is decreasing and between 2 and 5 it is increasing.

So it has a local minimum at -1 and local maximum at 2.

Also, the second derivative $-12x + 6$ is positive at -1 (minimum) and negative at 2 (maximum).