9/29/2015 Fall 2015, Calculus I Test I Review Sitaraman

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Instructions:
PLEASE PROVIDE STEP BY STEP EXPLANATIONS
WRITING ONLY ANSWERS WILL NOT GET FULL CREDIT
Time Limit 100 minutes
Please read the questions carefully before answering
Challenge problem is extra credit 20 points.
Total 100 points.

1. Given $f(x) = e^x$ and $g(x) = lnx$ find the following:

   (a) (4 points) $f \circ g(x)$ (b) (4 points) $g \circ f(x)$ (c) (2 points) $f \circ g(2)$

   Solution:

   Main point here is that the natural exponential and logarithm functions are inverses of each other, so they "cancel" each other out.

   (a) $f \circ g(x) = e^{ln(x)} = x$.

   (b) $g \circ f(x) = ln(e^x) = x$.

   (c) Using the answer for (a) we get $f \circ g(2) = 2$.

2. (a) (6 points) Show that $f(x) = \dfrac{x-3}{x^2-9}$ has a vertical asymptote at $x = -3$ by finding $\lim_{x\to -3} \dfrac{x-3}{x^2-9}$.

   (b) (12 points) Show that $f(x) = \dfrac{x-3}{x^2-9}$ has the $x-$axis as a horizontal asymptote by finding $\lim_{x\to -\infty} \dfrac{x-3}{x^2-9}$ and $\lim_{x\to \infty} \dfrac{x-3}{x^2-9}$.

   (c) (6 points) Find $\lim_{x\to 3} \dfrac{x-3}{x^2-9}$.

   (d) (6 points) Using (c), show that the function f has a removable discontinuity at 3 by defining $f(3)$ so that f is continuous at 3.

   Solution:

   2a.The vertical asymptotes are found by looking at the points where the denominator is zero. We have $x^2 - 9 = (x+3)(x-3) = 0$ means $x = -3$ or $x = 3$. But when $x = 3$ the numerator is also zero, so it is not a vertical asymptote. Canceling out $x - 3$ you get $f(x) = \frac{1}{x+3}$ when $x \neq 3$. As x approaches -3, the function approaches ∞ from the right and $-\infty$ from the left. So there is no limit, but the vertical line $x = -3$ is a vetical asymptote.

   2b. As $x \to \infty$, or as $x \to -\infty$, $f(x)$ goes to zero. To see this divide all by x^2. You get $f(x) = \dfrac{\frac{x}{x^2} - \frac{3}{x^2}}{\frac{x^2}{x^2} - \frac{9}{x^2}}$. In this expression everything goes to zero except x^2/x^2 which goes to 1. So the fraction as a whole goes to zero. Same is true when $x \to -\infty$. Thus we see that the $x-$axis is a horizontal asymptote because the function approaches the $x-$axis on both ends of the $x-$axis.

   2c. From (a) we have $\dfrac{x-3}{x^2-9} = \dfrac{x-3}{(x-3)(x+3)} = \dfrac{1}{x+3}$.

   Using this we get $\lim_{x\to 3} \dfrac{x-3}{x^2-9} = \lim_{x\to 3} \dfrac{1}{x+3} = 1/6$.

   2d. If we define $f(3) = 1/6$ then we have $\lim_{x\to 3} f(x) = f(3)$ and hence by definition of a continuous function, it becomes continuous at 3. Note that you cannot find $f(3)$ by plugging in 3 into $\frac{x-3}{x^2-9}$ because it beomes 0/0.

3. (20 points) Find the following limits:

$$(a) \lim_{x\to\infty} \frac{x^2+x+1}{2x^2-3x+15} \qquad (b) \lim_{x\to 0} \frac{\sqrt{x+1}-1}{x}$$

[Hint: For (b) multiply and divide by $\sqrt{x+1}+1$.]

Solution:

(a) Idea: Divide numerator and denominator by the highest power term namely x^2. You get:

$$Lim_{x\to\infty} \frac{\frac{x^2+x+1}{x^2}}{\frac{2x^2-3x+15}{x^2}} = Lim_{x\to\infty} \frac{1+\frac{1}{x}+\frac{1}{x^2}}{2-\frac{3}{x}+\frac{15}{x^2}} = \frac{1+0+0}{2-0+0} = 1/2.$$

(b) Idea: Multiply and divide by conjugate of numerator to get rid of square root in numerator (rationalizing the numerator).

$$\lim_{x\to 0} \frac{\sqrt{x+1}-1}{x} = \lim_{x\to 0} \frac{\sqrt{x+1}-1}{x} \frac{\sqrt{x+1}+1}{\sqrt{x+1}+1}$$

$$= \lim_{x\to 0} \frac{(\sqrt{x+1}-1)(\sqrt{x+1}+1)}{x(\sqrt{x+1}+1)} = \lim_{x\to 0} \frac{(\sqrt{x+1})^2-1^2}{x(\sqrt{x+1}+1)}$$

$$= \lim_{x\to 0} \frac{x+1-1}{x(\sqrt{x+1}+1)} = \lim_{x\to 0} \frac{x}{x(\sqrt{x+1}+1)} = \lim_{x\to 0} \frac{1}{\sqrt{x+1}+1} = 1/2.$$

4. (10 points) For $f(x) = x^3$ find $f'(2)$ using the basic definiton of derivative, namely $f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$.

(5 points) Also find the equation of the tangent at $x = 2$.

Solution:

$$\lim_{h\to 0} \frac{f(2+h)-f(2)}{h} = \lim_{h\to 0} \frac{(2+h)^3-2^3}{h} = \lim_{h\to 0} \frac{(2^3+3(2^2)h+3(2)h^2+h^3)-2^3}{h}$$

$$= \lim_{h\to 0} \frac{12h+6h^2+h^3}{h} = \lim_{h\to 0} 12+6h+h^2 = 12$$

The equation of the line passing through $(2, 2^3)$ or (2,8) with slope 12 is given by $y-8 = 12(x-2)$ or $y = 12x-16$.

5. Determine the average rate of change of the function $f(x) = x^3 + x$ over the interval $[1, 2]$. [This is same as the slope of the secant line from $x = 1$ to $x = 2$.]

Solution:

The average rate of change is given by

$$\frac{f(2)-f(1)}{2-1} = \frac{(2^3+2)-(1^3+1)}{1} = 8.$$

6. Find the limit $\lim_{x\to -2} \frac{x+2}{x^2-4}$.

Solution:

Since $x^2-4 = (x+2)(x-2)$, we get after canceling $x+2$ above and below

$$\lim_{x\to -2} \frac{x+2}{x^2-4} = \lim_{x\to -2} \frac{1}{x-2} = \frac{-1}{4}$$

7. If $5 - 2x^2 \leq f(x) \leq 5 - x^2$ for $-1 \leq x \leq 1$ find $\lim_{x\to 0} f(x)$. [Use squeeze theorem aka sandwich theorem]

   Solution:

   The function on the left, namely $5 - 2x^2$, goes to 5 as x goes to 0. This is easy to see because it is a polynomial function, so limit can be obtained by just plugging in 0.

   Similarly the function on the right also goes to 5.

   So by sandwich theorem, the function in the middle must also go to 5.

   So $\lim_{x\to 0} f(x) = 5$.

8. Let $f(x) = 2x + 1$, $L = \lim_{x\to 1} 2x + 1 = 3$, $\epsilon = 0.05$. Give a value for $\delta > 0$ such that for all x, $0 < |x - 1| < \delta$ implies $|f(x) - L| < \epsilon$.

   Solution:

   We want $|(2x + 1) - 3| < 0.05$. Need to solve for $|x - 1| < \delta$ such that this happens.

   Now,$|(2x + 1) - 3| < 0.05 \implies |2x - 2| < 0.05 \implies 2|x - 1| < 0.05 \implies |x - 1| < 0.05/2 = 0.025$.

   So if we choose $\delta = 0.025$ then for $0 < |x - 1| < 0.025$ we will have $|2x - 2| = 2|x - 1| < 0.05$.

9. Find the slope of the graph of $y = x^3$ at $x = 1$.

   Solution: We can do this just like we did it in class for $y = x^2$.

   For a point at $x = 1 + h$, the slope of the secant line from (1,1) to $(1 + h, (1 + h)^3)$ is given by

   $$\frac{(1+h)^3 - 1}{(1+h) - 1} = \frac{h^3 + 3h^2 + 3h}{h} = h^2 + 3h + 3.$$

   $$\text{So the slope is } \lim_{h\to 0}(h^2 + 3h + 3) = 3.$$

   The limit is found by just plugging in $h = 0$ because it is a polynomial.

10. Let

    $$f(x) = \begin{cases} 1 + x, & x < 1, \\ x^2 + 1, & x \geq 1. \end{cases}$$

    (10 points) Find $\lim\limits_{x\to 1^+} f(x)$ and $\lim\limits_{x\to 1^-} f(x)$. Does $\lim\limits_{x\to 1} f(x)$ exist? If so, what is it? If not, why not?

    (5 points) Where is the function $f(x)$ continuous and where is it discontinuous?

    Solution:

    To find $\lim\limits_{x\to 1^+} f(x)$ we need to look at what happens to the right of 1.

    Here the function is defined by $x^2 + 1$ which is a continuous function and approaches 2 as x approaches 1. [Since it is continous the value is obtaind by just plugging in 1]. So the limit is 2.

    To find $\lim\limits_{x\to 1^-} f(x)$ we need to look at what happens to the left of 1.

    Here the function is defined by $1 + x$ which is also a continuous function and approaches 2 as x approaches 2. As before, since it is continous the value is obtaind by just plugging in 1. So the limit is 1.

    Since the two limits are the same, $\lim\limits_{x\to 1} f(x)$ exists and also equals 2.

    From above, we see that it is continuous at 1. First note that $f(1) = 2$ because at $x = 1$ the function is defined by $x^2 + 1$. So the value of the function is same as the limit that it approaches as x approaches 1. Thus it is continuous at 1. Since it is defined by continuous functions $1 + x$ and $x^2 + 1$ respectively at all other points, it is continuous everywhere else also.

11. (10 points) An object is thrown down from 50 feet and its height (or position) after t seconds is given by $H(t) = 50 - 4.9t^2$. Find its velocity after 5 seconds. You must use the limit formula (see problem 3) to find the derivative.

Solution:

Velocity is given by the derivative of the height function.

Using the limit formula,

$$H'(5) = \lim_{h \to 0} \frac{H(5+h) - H(5)}{h} = \lim_{h \to 0} \frac{(50 - 4.9(5+h)^2) - (50 - 4.9(5^2))}{h}$$

$$= \lim_{h \to 0} \frac{(50 - 4.9(5^2 + 10h + h^2)) - (50 - 4.9(5^2))}{h} = \lim_{h \to 0} \frac{4.9(-10h - h^2)}{h} = \lim_{h \to 0} -49h - 4.9h = -49$$

12. Show that $\lim_{x \to \infty} \frac{x}{x+2} = 1$ using basic definition of limit. [Your answer should go something like this: We can make the difference between $x/(x+2)$ and 1 as small as we want by choosing x big enough. Start by producing an M such that when $x > M$ this happens]

Let ϵ be a very small real number. We want to show that the difference between $x/(x+2)$ and 1 is smaller than ϵ (*no matter how small it is*) if we choose x big enough.

So we want

$$\left| \frac{x}{x+2} - 1 \right| < \epsilon.$$

In other words

$$\left| \frac{x - (x+2)}{x+2} \right| < \epsilon.$$

i.e,

$$\left| \frac{-2}{x+2} \right| = \frac{2}{|x+2|} < \epsilon.$$

Here we can get rid of absolute value now because when $x > M$ where M is a big positive number $x + 2$ is also positive so everything is positive and the absolute value doesn't change anything. So we want $2/(x+2) < \epsilon \implies x + 2 > 2/\epsilon$. (Solved for $x + 2$). But now all we need to do is to choose $M = (2/\epsilon) - 2$ then if $x > M$ the above iequalities will all be satisfied and $x/x + 2$ will be within ϵ distance of 1.

13. Let

$$f(x) = \begin{cases} 3 - x, & x < 2, \\ x + 2, & x \geq 2. \end{cases}$$

(8 points) Find $\lim_{x \to 2^+} f(x)$ and $\lim_{x \to 2^-} f(x)$. Does $\lim_{x \to 2} f(x)$ exist? If so, what is it? If not, why not?

(6 points) Where is the function $f(x)$ continuous and where is it discontinuous?

Solution:

To find $\lim_{x \to 2^+} f(x)$ we need to look at what happens to the right of 2.

Here the function is defined by $x + 2$ which is a continuous function and approaches 4 as x approaches 2. [Since it is continous the value is obtaind by just plugging in 2]. So the limit is 4.

To find $\lim_{x \to 2^-} f(x)$ we need to look at what happens to the left of 2.

Here the function is defined by $3 - x$ which is also a continuous function and approaches 1 as x approaches 2. As before, since it is continous the value is obtaind by just plugging in 2. So the limit is 1.

Since the two limits are different, $\lim_{x \to 2} f(x)$ does not exist.

From above, we see that it is not continuous at 2 because the limit does not even exist there. But since it is defined by continuous functions $3 - x$ and $x + 2$ respectively at all other points, it is continuous everywhere else.

14. Imagine a baseball thrown at an angle along the $x-$axis from (0,0). At a distance of x from (0,0), the height is $y = 20x - x^2$. [i.e, the ball is at the point $(x, 100x - x^2)$]. The ground speed of the ball is given by 10 feet per second. In other words, after $t = 1$ second, $x = 5$ and after $t = 2$, $x = 10$ and in general, $x = 5t$. You must use derivatives to get credit.

    (a) Find the vertical velocity of the ball, namely $y'(t)$ [the rate of change of y with respect to t].

    (b) At what times is the ball going up and at what times is it going down?

    Solution: We $x = 5t$. Therefore, $y = 20(5t) - (5t^2) = 100t - 25t^2$. Thus,using limit formula, we get

    $$y'(t) = \lim_{h \to 0} \frac{y(t+h) - y(t)}{h} = \lim_{h \to 0} \frac{100(t+h) - 25(t+h)^2 - (100t - 25t^2)}{h} = 100 - 50t$$

    after simplification.

    It is going up when $y'(t) > 0$. This gives $100 - 50t > 0 \implies t < 2$.

    It is going down when $y'(t) < 0$. This gives $100 - 50t < 0 \implies t > 2$.

    Note that t is between 0 and 4 because $y \geq 0$ only in [0,4].

15. The Intermediate Value Theorem says that if the function $f(x)$ is continuous between $x = a$ and $x = b$ then it takes every value between $f(a)$ and $f(b)$. So for example, if $f(a)$ is negative and $f(b)$ is positive, then $f(x)$ will be zero (i.e, the graph wil cross the $x-$axis) for some x between a and b. Using this idea, show that $x^3 - 10x + 1$ equals zero (i.e, graph crosses the $x-$axis) three times between -4 and 4. [It is not enough just to draw a graph]

    Solution: We have $f(-4) = -23, f(-1) = 10, f(1) = -8, f(4) = 25$.

    So $f(x)$ goes from negative to positive, then back to negative and then again to positive.

    So it will become zero (graph will cross $x-$axis) between -4 and -1, then between -1 and 1 and again between 1 and 4.

16. Find the derivative of $f(x) = \frac{1}{x^{3/2}} - x^6 + e^2$.

    Solution:

    We have $f'(x) = (x^{-\frac{3}{2}} - x^6 + e^2)' = -\frac{3}{2}x^{-\frac{3}{2}-1} - 6x^5 + 0$.

    Derivative of e^2 is zero is because e^2 is a constant (just a number).

    Thus $f'(x) = \frac{-3}{2}x^{-\frac{5}{2}} - 6x^5 = \frac{-3}{2x^{5/2}} - 6x^5$.

17. A car is traveling along the x axis is at $x = 0.001t^2 + 0.005t$ miles after t seconds. Find the time at which it reaches 60mph.

    Solution: Velocity after t seconds is $x'(t) = 0.001(2t) + 0.005$ miles per second.

    In an hour there are 3600 seconds. So velocity reaches $3600(0.002t + 0.005) = 7.2t + 18$ miles per hour.

    If $60 = 7.2t + 18 \implies 7.2t = 60 - 18 = 42 \implies t = 42/(7.2) = 5.83$ seconds.

    Alternately, you can convert 60 mph to $60/3600 = 0.0167$ miles per second and set $0.0167 = 0.002t + 0.005$ and solve this to get the same answer.