10/2/2015 Fall 2015, Calculus I Test I Sitaraman

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Instructions:
PLEASE PROVIDE STEP BY STEP EXPLANATIONS
WRITING ONLY ANSWERS WILL NOT GET FULL CREDIT
Time Limit 50 minutes. Total 100 points.
Please read the questions carefully before answering

1. Given $f(x) = 10^x$ and $g(x) = logx$ find the following:

   (a) (6 points) $f \circ g(x)$ (b) (6 points) $g \circ f(x)$ (c) (3 points) $f \circ g(2)$

   Solution:

   Main point here is that the natural exponential and logarithm functions are inverses of each other, so they "cancel" each other out. Remember that $logx$ actually means logarithm to the base 10, namely $log_{10}x$.

   (a) $f \circ g(x) = 10^{log(x)} = x$.

   (b) $g \circ f(x) = log(10^x) = x$.

   (c) Using the answer for (a) we get $f \circ g(2) = 2$.

2. (20 points) Find the following limits, if they exist:

$$(a) \lim_{x\to\infty} \frac{x+1}{x^2-4} \qquad (b) \lim_{x\to 0} e^{\frac{1}{x}}$$

   Solution:

   (a) Idea: Divide numerator and denominator by the highest power term of *both of them* namely x^2. You get:

$$Lim_{x\to\infty} \frac{\frac{x+1}{x^2}}{\frac{x^2-4}{x^2}} = Lim_{x\to\infty} \frac{\frac{1}{x} + \frac{1}{x^2}}{1 - \frac{4}{x^2}} = \frac{0+0}{1-0} = 0$$

   If you divide numerator by x because that is the highest power in the numerator, then you are changing the value of the fraction because you are dividing by x and x^2. You need to divide *both by the same power* which is the highest power of all the terms, whether it be numerator or denominator.

   (b) Main point here is that as $x \to 0^+$, $1/x \to \infty$ and also that e^x goes to infinity as $x \to \infty$. So limit does not exist. Not enough to check for one or two points. You need to use the fact that e^x goes to infinity if the exponent goes to infinity.

   But note that as $x \to 0^-$, (from the negative side), $e^{1/x}$ goes to 0 because as $x \to 0^-$, we have $1/x \to -\infty$ and when the exponent goes to $-\infty$ the function e^u goes to 0. So while it is true that *as long as $f(x)$ does not approach a fiunite value, limit doesn't exist,* it is also not true in this case that $e^{1/x} \to \infty$. It goes to ∞ as you approach 0 from the right and approaches 0 as you approach from the left.

3. (10 points) For $f(x) = x^2$ find $f'(2)$ using the basic definiton of derivative, namely $f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$.

   (5 points) Also find the equation of the tangent at $x = 2$.

   Solution:

$$\lim_{h\to 0} \frac{f(2+h) - f(2)}{h} = \lim_{h\to 0} \frac{(2+h)^2 - 2^2}{h} = \lim_{h\to 0} \frac{(2^2 + 2(2)h + h^2) - 2^2}{h}$$

$$= \lim_{h\to 0} \frac{4h+h^2}{h} = \lim_{h\to 0} 4+h = 4$$

The equation of the line passing through $(2, 2^2)$ or (2,4) with slope 4 is given by $y - 4 = 4(x - 2)$ or $y = 4x - 4$.

4. (15 points) Find $\lim_{x\to 0} xsin(1/x)$. [Use squeeze theorem a.k.a sandwich theorem]

   Solution:

   $sin(1/x)$ varies between -1 and 1.

   So $xsin(1/x)$ is "squeezed" between $-x$ and x.

   As $x \to 0$, we have $-x \to 0$ also.

   So the functions on both sides go to 0 and by squeeze a.k.a sandwich theorem, the function in the middle must also go to 0.

   So $\lim_{x\to 0} xsin(1/x) = 0$.

5. (10 points) An object is thrown down from 100 feet and its height (or position) after t seconds is given by $H(t) = 100 - 4.9t^2$. Find its velocity and speed after 5 seconds.

   Solution:

   Velocity is given by the derivative of the height function. Speed is absolute value of velocity

   $H'(t) = -4.9(2t) = -9.8t; H'(5) = -9.8(5) = -49$ meters per second. Speed is 49 meters per second.

6. (15 points) Show that $\lim_{x\to\infty} \frac{x+2}{x} = 1$ using basic definition of limit. [Your answer should go something like this: We can make the difference between $(x + 2)/x$ and 1 as small as we want by choosing x big enough. Start by producing an M such that when $x > M$ the difference is as small as desired]

   Let ϵ be a very small real number. We want to show that the difference between $x/(x + 2)$ and 1 is smaller than ϵ (*no matter how small it is*) if we choose x big enough.

   So we want

   $$\left|\frac{x+2}{x} - 1\right| < \epsilon.$$

   In other words

   $$\left|\frac{x+2-x}{x}\right| < \epsilon.$$

   i.e,

   $$\left|\frac{2}{x}\right| < \epsilon.$$

   Here we can get rid of absolute value now because when $x > M$ where M is a big positive number x is also positive so everything is positive and the absolute value doesn't change anything. So we want $2/x < \epsilon \implies x > 2/\epsilon$. (Solved for x). But now all we need to do is to choose $M = (2/\epsilon)$ then if $x > M$ the above iequalities will all be satisfied and $(x + 2)/x$ will be within ϵ distance of 1.

7. (10 points) Let

   $$f(x) = \begin{cases} 1-2x, & x < 1, \\ x+2, & x \geq 1. \end{cases}$$

   Find $\lim_{x\to 1^+} f(x)$ and $\lim_{x\to 1^-} f(x)$. Does $\lim_{x\to 1} f(x)$ exist? If so, what is it? If not, why not?

   Is the function $f(x)$ continuous at $x = 1$?

Solution:

To find $\lim_{x \to 1^+} f(x)$ we need to look at what happens to the right of 1.

Here the function is defined by $x + 2$ which is a continuous function and approaches 3 as x approaches 1. [Since it is continous the value is obtaind by just plugging in 1]. So the limit is 3.

To find $\lim_{x \to 1^-} f(x)$ we need to look at what happens to the left of 1.

Here the function is defined by $1 - 2x$ which is also a continuous function and approaches -1 as x approaches 1. As before, since it is continous the value is obtaind by just plugging in 2. So the limit is -1.

Since the two limits are different, $\lim_{x \to 1} f(x)$ does not exist.

From above, we see that it is not continuous at 1 because the limit does not even exist there. [But since it is defined by continuous functions $1 - 2x$ and $x + 2$ respectively at all other points, it is continuous everywhere else].