12/17/2015 Fall 2015, Calculus I Final Exam brief Solutions Sitaraman

Howard University Math Department

1. Evaluate

   (a) $\lim_{x\to 0} \dfrac{e^x - x - 1}{x^2}$ (b) $\lim_{x\to\infty} \dfrac{2x}{\sqrt{x^2+x+2}}$

   Solution:

   Part (a) is best done using L'Hospital's rule while (b) is best done without it. For both you need to state why the rule applies, including whether derivative of denominator is not zero near where the limit is found.

   Actually for (a) need to apply L'Hospital's rule twice. Show that it goes to 0/0 both times.

$$\lim_{x\to 0} \frac{e^x - x - 1}{x^2} = \lim_{x\to 0} \frac{(e^x - x - 1)'}{(x^2)'} = \lim_{x\to 0} \frac{e^x - 1}{2x} = \lim_{x\to 0} \frac{(e^x - 1)'}{(2x)'} = \lim_{x\to 0} \frac{e^x}{2} = e^0/2 = 1/2.$$

   For part (b) divide above and below by x (NOT x^2). In the denominator write x as $\sqrt{x^2}$ so that you can take it inside the square-root.

$$\lim_{x\to\infty} \frac{2x}{\sqrt{x^2+x+2}} = \lim_{x\to\infty} \frac{2x/x}{\sqrt{(x^2+x+2)/x^2}} = \lim_{x\to\infty} \frac{2}{\sqrt{(x^2/x^2)+(x/x^2)+(2/x^2)}} = \lim_{x\to\infty} \frac{2}{\sqrt{1+(1/x)+(2/x^2)}}$$

$$= \lim_{x\to\infty} \frac{2}{\sqrt{1+0+0}} = 2/1 = 2.$$

2. Differentiate the following:

   (a) $f(x) = ln\left(\dfrac{e^{2x}}{x+1}\right)$.

   (b) $\cos(\sin(2x+1))$

   (c) e^{tanx}

   (d) $G(x) = \displaystyle\int_x^1 e^{t^2}\,dt$

   Solution:

   2a. This is best done by first simplifying the function using properties of logarithms.

   Write $ln\left(\dfrac{e^{2x}}{x+1}\right) = ln(e^{2x}) - ln(x+1) = 2x - ln(x+1)$.

   Now the derivative is just $2 - \dfrac{1}{x+1} = \dfrac{2x+1}{x+1}$.

   Many of you used chain rule first, to write the derivative of $ln(\ )$ as $1/(\ )$ and then differentiate what is inside. That is okay.You will then need to use quotient rule to differentiate what is inside.

   2b. Use chain rule three times here.

   $\cos(\sin(2x+1))' = -\sin(\sin(2x+1))(\sin(2x+1))' = -\sin(\sin(2x+1))(\cos(2x+1))(2x+1)' = -\sin(\sin(2x+1))(\cos(2x+1))(2)$.

   2c. Chain rule just once. Answer is $e^{\tan x}(\tan x)' = e^{\tan x}\sec^2 x$.

   2d. First write $G(x) = -\int_1^x e^{t^2}\,dt$ (when order of integration changes, sign of integral changes). Now apply fundamental theorem of calculus to get $G'(x) = -e^{x^2}$. Note that the function has x as the variable.

3.

$$\text{Given that } y \text{ is differentiable and } xy = x - e^2 + y^2 \text{ find } \frac{dy}{dx}$$

Solution:

Need to use product rule on left. Also remember that e^2 is just a number, and its derivative is zero. $xy' + y = 1 + 2yy'$. Solve for y' to get $y' = \frac{1-y}{x-2y}$

4. (40 points) Evaluate the following integrals.

(a) $\int \left(\sin x - \frac{1}{x} + e^2 \right)\ dx$

(b) $\int_0^1 (3x + \sqrt{x})\ dx$

(c) $\int \frac{ln(x^2)}{x}\ dx$

(d) $\int_0^2 xe^{-x^2}\ dx$

(e) $\int_{-3}^{3} \left(\sqrt{9-x^2}\right)\ dx$(Hint: Consider the graph of $y = \sqrt{9-x^2}$)

Solution:

For (a) note that e^2 is a constant, so its anti-derivative (integral) is e^2x.

Separating the integrals for each term in the sum, we get

$\int \left(\sin x - \frac{1}{x} + e^2 \right)\ dx = -\cos x - lnx + e^2x + C.$

For (b) use the formula $\int x^n = \frac{x^{n+1}}{n+1}$.

We get $\int_0^1 (3x + \sqrt{x})\ dx = \left[3\frac{x^2}{2} + \frac{x^{3/2}}{3/2} \right]_0^1 = \frac{3}{2} + \frac{2}{3} = 13/6.$

For (c) first write $ln(x^2) = 2lnx$ and then put $u = lnx$ to get $\int \frac{ln(x^2)}{x}\ dx = 2\int \frac{lnx}{x}\ dx = 2\int u\ du = 2u^2/2 + C = u^2 + C = (lnx)^2 + C$. Otherwise this problem is a bit tricky although many of you got it by setting $u = ln(x^2)$.

For (d) let $u = x^2$. Be careful with the limits. Either change it when you write integral in terms of u or keep them as it is until the integral is done and written in terms of x.

We get $\int_0^2 xe^{-x^2}\ dx = \int_{x=0}^{x=2} (du/2)e^{-u} = \left[-e^{-u}/2\right]_{x=0}^{x=2}$

$= \left[-e^{-x^2}/2\right]_0^2 = -e^{-4}/2 - (-e^0/2) = 1/2 - e^{-4}/2.$

For (e) note that the integral is the area under the semi-circle above $x-$axis with center at (0,0) and radius 3, with diameter going from (-3,0) to (0,3).

To see this set $y = \sqrt{9-x^2}$ and get $y^2 = 9 - x^2 \implies x^2 + y^2 = 9$. This is the equation of a circle with center at (0,0) and radius 3.

So the integral = area of semicircle of radius 3 $= \frac{\pi(3^2)}{2} = 4.5\pi = 14.137$ approximately.

5. A right cylindrical tank is filled with water. The tank stands upright and has a radius of 20 cm. How fast does the height of water in the tank drop when the water is being drained at 25 cubic-cms/sec?

   Solution:

   Note that only the height of the cylinder will be changing here. The radius is the same (constant) at all points of the cylinder. The volume of a cylinder is $V(h) = \pi r^2 h$. (Many of you used 1/3 of this –but that is the volume of a cone). If water is being drained at 25 cc/sec then $dV/dt = -25$. Note the negative sign – Volume is decreasing. So we get

   $$dV/dt = (dV/dh)(dh/dt) \implies -25 = (\pi r^2)(dh/dt) \implies dh/dt = \frac{-25}{\pi r^2} = \frac{-25}{\pi(20)^2} = -1/(16\pi) = -0.0199 cm/sec$$

6. Let $f(x) = \begin{cases} x - x^2 & if\ 0 \le x \le 2 \\ 2 - x, & if\ 2 \le x \le 3 \\ x - 4 & if\ x > 3 \end{cases}$

   (a) Find $\lim_{x \to 2^-} f(x)$

   (b) Find $\lim_{x \to 2^+} f(x)$

   (c) Find $\lim_{x \to 2} f(x)$, if it exists. Explain your answer.

   Solution:

   For (a) use $x - x^2$ because you need to approach 2 from left. The limit equals $2 - 2^2 = -2$ because $x - x^2$ is a continuous function and you can just plug in the value to find the limit.

   For (b) use $2 - x$ and as in (a) get the limit as $2 - 2 = 0$.

   For (c) the answer is that the limit does not exist because the left and right limits are not equal.

7. Find the absolute maximum and absolute minimum values of $f(x) = \frac{x^4}{4} + \frac{x^3}{3} - x^2 + 1$ on [-1, 2].

   Solution:

   The derivative is $f'(x) = x^3 + x^2 - 2x$.

   Critical points: There are no points where f is undefined. Only need to find where $f'(x) = 0$.

   Solving, we get $x^3 + x^2 - 2x = 0 \implies x(x^2 + x - 2) = 0 \implies x(x + 2)(x - 1) = 0$.

   The critical points are $x = 0, x = -2, x = 1$. Of these only 1 and 0 are in [-1,2].

   Now we only need to check the values of the function at 1, 0 and the boundary points -1 and 2.

   $f(-1) = (1/4) + (-1/3) - 1 + 1 = -1/12$. $f(0) = 0 + 0 - 0 + 1 = 1$. $f(1) = (1/4) + (1/3) - 1 + 1 = 7/12$. $f(2) = 4 + (8/3) - 4 + 1 = 11/3$.

   Comparing these values, we see that -1/12 is the absolute minimum, attained at $x = -1$.

   11/3 is the absolute maximum, attained at $x = 2$.

8. A rectangular box is to be made from a piece of cardboard 24 inches long and 9 inches wide by cutting out identical squares from the four corners and turning up the sides. Find the dimensions of the box of maximum volume.

   Solution:

   Let x be the length of the squares cut out from each corner. Then the dimensions of the box will be $l = 24 - 2x, w = 9 - 2x, h = x$. So volume is $V(x) = (24 - 2x)(9 - 2x)x$.

   This function needs to be maximized. It is well defined everywhere, so critical points are just those points where $V'(x) = 0$.

Using product rule, we find $V'(x) = -2(9-2x)x + (24-2x)(-2)x + (24-2x)(9-2x)(1) = -2(9x - 2x^2 + 24x - 2x^2) + (4x^2 - 66x + 216) = 12x^2 - 132x + 216$.

When $V'(x) = 0$, we have $12(x^2 - 11x + 18) = 0 \implies 12(x-9)(x-2) = 0$.

So the critical points are 9 and 2. Note that the boundary points are 0 and 4.5 because x can neither be negative nor bigger than half the width (or else it won't be possible to cut x amount from the corners).

So we just need to look at $x = 2$. $V(2) = 200cc$.

How do we know this is the maximum? At the boundary points the volume is zero and since 200 is bigger than zero it has to be the absolute maximum.

But we can also do this using derivatives and that is always recommended.

Check second drivative: $V''(x) = 24x - 132 = -84$ when $x = 2$.

Since second derivative is negative it is a local maximum, and since the boundary points result in a box of zero volume obviously, this is the absolute maximum.

9. Let $f(x) = \frac{x^3 - 2x}{x}$. Find an equation of the line that is tangent to the graph of $f(x)$ and parallel to the line $y - 4x + 7 = 0$.

   Solution: This problem is also easier if you simplify it first. Lot of people did this using quotient rule but that is not necessary.

   Write $f(x) = x^2 - 2$ by dividing x. Then $f'(x) = 2x$.

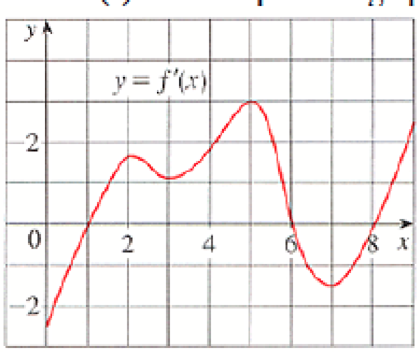
   We want the slope pf the tangent, i.e, the derivative, to equal the slope of $y - 4x + 7 = 0$. The slope of $y - 4x + 7 = 0$ is 4. So we want $2x = 4 \implies x = 2$. At this point the tangent of the graph is parallel to the given line and the slope of both equal 4.

   The equation of the line is $y = 4x + b$. When $x = 2, y = f(x) = 2^2 - 2 = 2$. Plugging this in we get $2 = 4(2) + b \implies b = -6$.

   So the desired line has equation $y = 4x - 6$.

10. The graph of the derivative of a continuous function is shown below.
20

   (a) Indicate the intervals where the function is increasing and the intervals where the function is decreasing.
   (b) At what values of *x* does *f* have a local maximum and minimum?
   (c) Indicate the intervals where the function is concave upward and the intervals where the function is concave downward.
   (d) State the *x*-coordinate(s) of the point(s) of inflection.
   (e) Sketch a possible graph of *f*.



Solution:

Note that the given graph is that of the derivative! Many people neglected that fact.

(a) The function is increasing when the derivative is positive and decreasing when it is negative. From the graph, the derivative is positive in (1,6) and $(8, \infty)$. So f is increasing there. The derivative is negative in $(-\infty, 1)$ and (6,8). So f is decreasing there.

(b) Local maximum or minimum when $f' = 0$. When $f' = 0$ graph of f' will cross $x-$axis. This happens at 1, 6 and 8.

(c) Concave up when $f'' > 0$. Since second derivative is derivative of derivative, this happens when derivative is increasing. From the graph this is in $(-\infty, 2), (3, 5), (7, \infty)$.

Concave down is when $f'' < 0$ or when derivative is decreasing. From graph this is in $(2, 3), (5, 7)$.

(d) Point of inflection is when $f'' = 0$. This means derivative of f' is zero. This happens at a local maximum or minimum of f'. The local maxima, minima of f' happen at 2, 3, 5, 7. These are the points of inflection.

(e) To draw a graph of this function use the data from a,b,c,d above. Note that at point of inflection the concavity of graph must change from concave up to down or vice versa. It is hard to tell where the graph will cross $x-$axis, if at all, just from knowing the derivative. So you just need to get the shape right. Position of graph doesn't matter as long as the local maxima or minima happen at 1,6 and 8.

Here is a sample graph, drawn approximately conforming to above data. Check that it does!