Instructions:
no calculators
PLEASE PROVIDE STEP BY STEP EXPLANATIONS
WRITING ONLY ANSWERS WILL NOT GET FULL CREDIT
Time Limit 45 minutes
Please read the questions carefully before answering
Each problem 20 points unless otherwise stated.
Any points you get in excess of 100 is extra credit.

1. Simplify and write with positive exponents (10 points each):

   (a) $\left(2^{20}x^{10}y^{25}\right)\left(8x^3y^{-20}\right)$ (b) $\dfrac{\left(x^{2/3}y^{3/4}\right)^2}{x^{1/3}y^{5/2}}$

   Soln:

   $$\text{1a. } 2^{20}x^{10}y^{25}(8x^3y^{-20}) = 2^{20}(8)x^{10}x^3y^{25}y^{-20} = (2^{20}2^3)x^{10+3}y^{25-20} = 2^{23}x^{13}y^5$$

   $$\text{1b. } \frac{\left(x^{2/3}y^{3/4}\right)^2}{x^{1/3}y^{5/2}} = \frac{\left(x^{2/3}\right)^2\left(y^{3/4}\right)^2}{x^{1/3}y^{5/2}} = \frac{\left(x^{4/3}y^{6/4}\right)}{x^{1/3}y^{5/2}} = x^{\frac{4}{3}-\frac{1}{3}}y^{\frac{3}{2}-\frac{5}{2}} = x^1y^{-1} = \frac{x}{y}$$

   Notice how we used the following rules:

   When a power is raised to another power, you multiply the powers

   When you multiply powers of same base, you add the exponents

   Notice how we wrote xy^{-1} as x/y so that all exponents will be positive.

   Also notice how we wrote 8 as 2^3. This is so we can combine2^{20} with 8.

   1b. $\dfrac{(81x^4)^{3/4}}{x^5} = \dfrac{(3^4)^{3/4}(x^4)^{3/4}}{x^5} = \dfrac{3^3x^3}{x^5} = 27x^{3-5} = 27x^{-2} = \dfrac{27}{x^2}$

2. (a) (10 points) Write the interval (-1,10] as an inequality involving x.

   (b) (10 points) The speed of light is exactly 2.99792458×10^8 meters/sec. Write this as a natural number.

   Soln:

   (a) $-1 < x \leq 10$.

   (b) Move the decimal 8 places to the right because you have a positive power.

   Answer: 299792458

3. Factor completely: $x^4 - 81$.

   Soln: $x^4 - 81 = (x^2)^2 - 9^2 = (x^2-9)(x^2+9) = (x^2-3^2)(x^2+9) = (x-3)(x+3)(x^2+9)$.

   $= (x+1)(x+1)(x-1) = (x+1)^2(x-1)$.

   We used the difference of squares formula $A^2 - B^2 = (A-B)(A+B)$ twice.

4. Say if true or false. Explain why.

(10 points each) (a) $(x^2+4)^2 = x^4+16$ (b) $100000^3 - 1^3 = 99999^3$.

Soln:

Both are false.

(a) $(x^2+4)^2 = (x^2)^2 + 2(x^2)(4) + 4^2 = x^4 + 8x^2 + 16$ using $(A+B)^2 = A^2 + 2AB + B^2$. Check: $(1^2+4)^2 = (1+4)^2 = 25$ but $1^4 + 16 = 1 + 16 = 17$. Note that even if it fails in one case the identity is not true.

Using $(A^3 - B^3) = (A-B)(A^2+AB+B^2)$ formula we get $100000^3 - 1^3 = 99999(100000^2 + 100000 + 1)$ which is bigger than 99999^3.

If you said both are false because we can separate the powers only when multiplication is involved, as shown in class, that is fine too.

5. Factor everything and simplify as much as possible:

$$\frac{3x^3 - 5x^2 - 50x}{x^2 - 7x + 10}.$$

Soln:

First we factor the denominator: $x^2 - 7x + 10 = (x-5)(x-2)$.

Next we factor the numerator by first taking out the common term x and then factoring $3x^2 - 5x - 50$ by figuring out that -5= -15+10 while 3(-50) = -150 = (-15)(10).

$$3x^3 - 5x^2 - 50x = x(3x^2 - 5x - 50) = x(3x^2 - 15x + 10x - 50) = x((3x(x-5) + 10(x-5)) = x(3x+10)(x-5).$$

Putting everything together we get

$$\frac{3x^3 - 5x^2 - 50x}{x^2 - 7x + 10} = \frac{x(3x+10)(x-5)}{(x-5)(x-2)} = \frac{x(3x+10)}{x-2}.$$

6. (Optional, 20 points extra credit) Prove that $\left(\sqrt{7} - \sqrt{6}\right)^{-1} = \sqrt{7} + \sqrt{6}$. i.e, the reciprocal of $\sqrt{7} - \sqrt{6}$ equals $\sqrt{7} + \sqrt{6}$. Show that in general $\sqrt{n+1} + \sqrt{n}$ is the reciprocal of $\sqrt{n+1} - \sqrt{n}$ for any natural number n.

Soln: This can be done by rationalizing the denominator.

$$\frac{1}{\sqrt{7} - \sqrt{6}} = \frac{1}{(\sqrt{7} - \sqrt{6})} \frac{(\sqrt{7} + \sqrt{6})}{(\sqrt{7} + \sqrt{6})} = \frac{\sqrt{7} + \sqrt{6}}{(\sqrt{7})^2 - (\sqrt{6})^2} = \frac{\sqrt{7} + \sqrt{6}}{7 - 6} = \sqrt{7} + \sqrt{6}.$$

The last part is done in exactly same way. Simply replace 7 with $n+1$ and 6 with n. You can convince yourselves by trying n=1,2,3, etc., and noticing that in all these cases the argument is the same. You multiply above and below by $\sqrt{2} + \sqrt{1}$ in the case of $n = 1$ and $\sqrt{n+1} - \sqrt{n} = \sqrt{2} - \sqrt{1}$ and then by $\sqrt{3} + \sqrt{2}$ when $n = 2$ then by $\sqrt{4} + \sqrt{3}$ and so on.