4/22/09 Spring 09, Calculus II Test IV solutions Sitaraman

1. (10 pts) Convert the following to polar co-ordinates: $xy = 1$.

soln: $(rcos\theta)(rsin\theta) = 1$ or $r^2 = sec\theta cosec\theta$.

2. (15 pts) Sketch the curve in polar co-ordinates by marking the graph at $\theta = 0, \pi/3, \pi/6, \pi/2, 2\pi/3, \pi$ and using symmetry: $r = 2 + sin\theta$. Say what kind of curve it is.

This is a convex limacon. Graph is on website under this file.

3.(15 pts) Find the slope of the tangent line to the polar curve $r = 1 - cos\theta$ at $\theta = \pi/2$.

Soln: The slope is found using the formula $\frac{rcos\theta + (dr/d\theta)sin\theta}{-rsin\theta + (dr/d\theta)cos\theta}$ and setting $\theta = \pi/2$. Note that $dr/d\theta = sin\theta$. Answer = -1.

4. (15 pts) Find the area of the region enclosed by the circle $r = 2cos\theta$ using integration over θ.

soln: Area is given by $2\int_0^{\frac{\pi}{2}}(1/2)r^2 d\theta$ by symmetry.

This is $2\int_0^{\frac{\pi}{2}}(1/2)4cos^2\theta d\theta = 2\int(1 + cos(2\theta))d\theta = \pi$. Note that the circle is traced fully when the angle varies from 0 to π and the semicircle is traced when the angle goes from 0 to $\pi/2$.

5. (15 pts) Find the length of the arc of the curve $r = 2sec\theta$ from $\theta = 0$ to $\theta = \pi/4$.

Soln: length of arc is given by $\int_0^{\frac{\pi}{4}}\sqrt{r^2 + (\frac{dr}{d\theta})^2}\ d\theta$. Putting $r = 2sec\theta$ and $dr/d\theta = 2sec\theta tan\theta$ and simplifying we get the length as $L = 2\int_0^{\frac{\pi}{2}} sec^2\theta d\theta = 2$.

6. (15 pts) Sketch the ellipse with equation $16x^2 + 25y^2 = 400$ and label the vertices, foci, and the ends of the minor axis.

soln: dividing throughout by 400, we get $\frac{x^2}{25} + \frac{y^2}{16} = 1$ which is an ellipse with major axis along the x-direction, $a = 5, b = 4, c = 3$. Foci are at $(3, 0)$ and $(-3, 0)$, the vertices are at $(5, 0), (-5, 0)$ and the ends of the minor axis are at $(0, 4)$ and $(0, -4)$.

7. (15 pts) Sketch the hyperbola with equation $xy = 1$ by finding its equation after rotating the co-ordinate axes and label the vertices and foci.

Soln: A=C=0. So the angle of rotation is given by $(A - C)/B = 0 = cot2\theta$ which means $\theta = \pi/4$. So the rotation equation is given by $x = x'cos\theta - y'sin\theta = (x' - y')/\sqrt{2}$ and $y = x'sin\theta + y'cos\theta = (x' + y')/\sqrt{2}$. Plugging into the original equation and simplifying we get $\frac{(x')^2}{2} - \frac{(y')^2}{2} = 1$ from which we get a hyperbola along the line $x = y$ which is at an agle of 45 degrees to the horizontal, with $a = b = \sqrt{2}$ and $c = \sqrt{a^2 + b^2} = 2$. So the vertices are at a distance of $\sqrt{2}$ units from the origin along this diagonal line on either side of the origin and the foci are at a distance of 2 along this diagonal line on either side of the origin.

8. [Challenge problem, 20 points] The planets' orbits around the sun is an ellipse, with the sun at one of the foci. We can write the equation of the ellipse in the form $r = \frac{ed}{1+ecos\theta}$ where d and e are constants and the pole is at one of the foci (in this case we can take that as the focus where the sun is) and the polar axis is also the major axis. Find the farthest and closest that the planet is to the sun, in terms of d and e.

Soln: the farthest and closest points are obtained by setting θ to equal π and 0 respectively in this equation. So you get $ed/(1 - e)$ and $ed/(1 + e)$ respectively.