4/3/09 Spring 09, Calculus II Test III solutions Sitaraman

Instructions:
PLEASE PROVIDE STEP BY STEP EXPLANATIONS
ANSWERS WITHOUT EXPLANATION WILL ONLY GET 40%
Time Limit 45 minutes
Please read the questions carefully before answering
It is recommended that you try those problems you are most comfortable with, first.
Attempt as many as you can; Anything over 100 is extra credit.

1. (10 pts) Find the general term of the following **sequence** starting with $n = 1$, determine if it converges and if it does find limit: $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots$

Soln: General term is $\frac{n}{n+1}$. The sequence has the limit 0 as n goes to ∞. To see this write $\frac{n}{n+1}$ as $1 - \frac{1}{n+1}$.

2. (18 pts).State if true or false. Prove your answer if possible ; else provide counterexample:

a) If a_n goes to 0 as n goes to ∞, then Σa_n converges. (b) If $a_k = f(k)$ for $k \geq 1$ and $f(x)$ is a positive, continuous and decreasing function for $x \geq 1$, then $\sum_{k=1}^{\infty} a_k = \int_1^\infty f(x)\, dx$. (c) If the function $f(x) = \sum_{k=0}^{\infty} a_k x^k$, then $a_k = \frac{f^{(k)}(0)}{k!}$ in the interval where the series converges.

Soln:

(a): False. Counterexample: The sequence $1/n$ goes to 0 but the series $\Sigma 1/n$ diverges.

(b) False. The integral can be used to check whether the series converges, but their sums are not equal. Counterexample:

$\sum_{k=1}^{\infty} \frac{1}{e^k} = \frac{1}{e-1}$ but $\int_1^\infty \frac{dx}{e^x} = 1/e$.

(c) True. Theorem 10.10.6

3.(20 pts) Find the sum of the following: (a) $\sum_{k=1}^{\infty} \frac{2}{3^k}$ (b) $\sum_{k=1}^{\infty} \frac{5}{2k(2k+2)}$

Soln: (a) This is a geometric series with first term $a = 2/3$ and common ratio $r = 1/3$ and the sum is $\frac{2/3}{1-(1/3)} = 1$. (b) This is a telescoping series with first term $5/4$ and so the sum is also $5/4$ because

$s_n = \sum_{k=1}^{n} \frac{5}{2k(2k+2)} = \sum_{k=1}^{n} \frac{5}{2}\left(\frac{1}{2k} - \frac{1}{2k+2}\right) = 5/2(1/2 - 1/4 + 1/4 - 1/6 + \ldots 1/(2n) - 1/(2n+2)) = 5/4 - (5/(2(2n+2))$ and the second term goes to 0 as n goes to ∞.

4. (20 pts) Check for convergence by any method: (a) $\sum_{k=1}^{\infty} \frac{k}{lnk}$ (b) $\sum_{k=1}^{\infty} \frac{e^2}{k^{\sqrt{2}}}$

Soln: (a) This diverges because k/lnk does not go to 0 as k goes to ∞. To see this use L'Hospital's rule on x/lnx. (b) This converges by the $p-$test because e^2 is just a constant and the series $\Sigma(1/k^{\sqrt{2}})$ is a peries with $p = \sqrt{2}$ which is greater than 1.

5. (12 pts) (a) Check for convergence using comparison test: $\sum_{k=1}^{\infty} \frac{1}{k^2+2}$

(b) Check for convergence using ratio test: $\sum_{n=1}^{\infty} \frac{n!}{3^n}$

(a) This converges because $\frac{1}{k^2+2} < \frac{1}{k^2}$ and $\Sigma 1/k^2$ converges by the p-test. (b) Using ratio test, $\rho = \lim_{n\to\infty} (a_{n+1}/a_n) = \lim_{n\to\infty} \frac{(n+1)!}{3^{n+1}} / \frac{n!}{3^n} = \lim_{n\to\infty} (n+1)/3 = \infty$. So this series diverges.

6. (20 pts) Find the Maclaurin series of $cosx$ (write first three terms and general term) using its derivatives. Find the radius of convergence.Use the Maclaurin series to evaluate the sum $1 - \frac{1}{2!} + \frac{1}{4!} - \frac{1}{6!} + \ldots$

Solution: $f(x) = cosx$. $f(0) = 1$, $f'(0) = -sin(0) = 0$, $f''(0) = -cos(0) = -1$, $f^{(3)}(0) = sin(0) = 0$, $f^{(4)}(0) = cos(0) = 1, \ldots$ $f^{(k)}(0) = 0$ if k is odd and $f^{(2n)}(0) = -1$ if n is odd and 1 if n is even (so actually $f^{(2)}(0) = f^{(6)}(0)\ldots = -1$ and $f^{(4)}(0) = f^{(8)}(0)\ldots = 1$. Thus the Maclaurin series is $cosx = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots + (-1)^n \frac{x^{2n}}{2n!} + \ldots$. The radius of convergence is found using ratio test for absolute convergence. The absolute value of ratio of $n+1$th term to $n-$th term is $\frac{x^2}{2n+2}$ and this goes to zero regardless of the value of x. So radius of convergence is ∞. To find the above sum you just put $x = 1$ in the expansion for $cosx$ and get the answer as $cos1 = 0.5403$.

NOTE: THERE WAS A TYPO IN THIS PROBLEM. HAVE GIVEN GENEROUS PARTIAL CREDIT. THE WAY IT WAS GIVEN IN TEST BOTH SERIES DIVERGE BY P-TEST. NOTE CORRECTED VERSION BELOW.

7. [Challenge problem, 30 points] (a)Show: Only one of $\sum_{n=1}^{\infty} \frac{cos^2 n\theta}{n}$ or $\sum_{n=1}^{\infty} \frac{sin^2 n\theta}{n}$ can converge.(b) Now, it is known that $\sum_{n=1}^{\infty} \frac{cos 2n\theta}{n}$ converges. Using this, show that $\sum_{n=1}^{\infty} \frac{cos^2 n\theta}{n}$ diverges. [Don't use (b) to prove (a)].

Soln: For part (a) we can use the identity $cos^2 n\theta + sin^2 n\theta = 1$. If both series converge, then their sum must also converge. But their sum is $\sum_{n=1}^{\infty} \frac{cos^2 n\theta}{n} + \sum_{n=1}^{\infty} \frac{sin^2 n\theta}{n} = \sum_{n=1}^{\infty} \frac{1}{n}$ which diverges. So at the most only one of them can converge. For (b) use the identity $cos(2n\theta) = 2cos^2 n\theta - 1$ to get $2cos^2 n\theta - cos(2n\theta) = 1$. Using the same argument as in (a) we get that the difference of two convergent series $2\sum_{n=1}^{\infty} \frac{cos^2 n\theta}{n}$ and $\sum_{n=1}^{\infty} \frac{cos(2n\theta)}{n}$ is divergent. [Note that using the identity $cos(2n\theta) = 1 - 2sin^2 n\theta$ we can prove that $\sum_{n=1}^{\infty} \frac{sin^2 n\theta}{n}$ also diverges].