Spring 10, Calculus I Test II Howard University Mathematics
Math 156-05 Sitaraman 3/5/2010

Instructions:
PLEASE PROVIDE STEP BY STEP EXPLANATIONS
Time Limit 45 minutes
Please read the questions carefully before answering
Each problem 10 points unless otherwise stated.
Total 120 points including 20 points extra credit for challenge problem.

1. Given that $\underset{x\to\infty}{Lim}\frac{1}{e^{x^{0.1}}} = 0$, and $\epsilon = 0.001$, find N such that $x > N$ means $|f(x) - L| < \epsilon$ where $f(x) = \frac{1}{e^{x^{0.1}}}$ and $L = 0$.

Soln: We need $|\frac{1}{e^{x^{0.1}}}| < 0.001$ which means $|e^{x^{0.1}}| > 1000$. Since we are really loking for positive x, we can let $|e^{x^{0.1}}| = e^{x^{0.1}}$. So we get $e^{x^{0.1}} > 1000$ so $x^{0.1} > ln(1000)$ and $x > (ln(1000))^{10} = 247382762$.

2. Find all the points where $e^{\frac{1}{x}}$ is continuous.

Soln: $1/x$ is continuous everywhere except 0. Since e^x is continuous everywhere, the composition is continuous at all points except 0. Note that $e^{1/x}$ approaches 0 as x approaches 0 from the left! But from the right it goes to infinity.

3. (20 points) Let the function $y = d(t) = 2t - t^2$ give the distance travelled by an object in t seconds. The rate of change of distance with respect to time is the velocity of the object. Find the following: (a) average rate of change (average velocity) from $t = 0$ to 0.1. (b) average rate of change from $t = 0$ to 0.01. (c) Instantaneous rate of change at $t = 0$ (velocity at $t = 0$) using the limit formula for derivative.

Soln: (a) Average rate of change $= \frac{f(0.1)-f(0)}{0.1-0} = \frac{2(0.1)-0.1^2-0}{0.1} = 1.9$.

(b) Average rate of change $= \frac{f(0.01)-f(0)}{0.01-0} = \frac{2(0.01)-0.01^2-0}{0.01} = 1.99$

(c) Using limit formula the instantaneous rate of change is $d'(t) = \underset{h\to 0}{Lim}\frac{d(t+h)-d(t)}{h} = \underset{h\to 0}{Lim}\frac{2(t+h)-(t+h)^2-(2t-t^2)}{h} = \underset{h\to 0}{Lim}\ 2 - 2t - h = 2 - 2t$. Setting $t = 0$ we get $d'(0) = 2 - 2(0) = 2$.

4. Find the velocity at 0 (i.e, $d'(0)$) of the object in problem 3 using derivatives.

Soln: $d'(t) = (2t - t^2)' = 2 - 2t$. So $d'(0) = 2 - 2(0) = 2$.

5. (20 points) Find the equation of the tangent line to the graph of $f(x)$ where $f(x) = y = \frac{x-1}{x+1}$ at a general point $x = a$. Find the points where it has a horizontal tangent line or the slope of the tangent is undefined.

Soln: Slope of the tangent at $x = a$ is $f'(a)$. We find $f'(x)$ using quotient rule: $f'(x) = \frac{(x-1)'(x+1)-(x-1)(x+1)'}{(x+1)^2} = \frac{2}{(x+1)^2}$. So the slope at $x = a$ is $2/(a+1)^2$ and the equation is $y - f(a) = \frac{2}{(a+1)^2}(x - a)$ where $f(a) = (a-1)/(a+1)$. The slope is never zero because numerator is 2 and hence never equal to 0. It is undefined when $a = -1$.

6. Find the derivative of $f(x) = (x^2 + x + 1)(x^3 - 2x)$ using product rule.

Soln: $f'(x) = (x^2 + x + 1)'(x^3 - 2x) + (x^2 + x + 1)(x^3 - 2x)' = (2x+1)(x^3 - 2x) + (x^2 + x + 1)(3x^2 - 2) = 2x^4 - 4x^2 + x^3 - 2x + 3x^4 - 2x^2 + 3x^3 - 2x + 3x^2 - 2 = 5x^4 + 4x^3 - 3x^2 - 4x - 2$.

7.Given that $f(1) = 1$, $f'(1) = -1$, $g(1) = 2$, and $g'(1) = 1$, find the derivative of $f(x)g(x)$ at $x = 1$.

Soln: $(fg)' = f'g + g'f$. So $fg'(1) = f'(1)g(1) + g'(1)f(1) = (-1)(2) + (1)(1) = -1$.

8. Find the derivative of $sin(x^2)$ and $(x^2 + 1)^{10}$ using chain rule.

Soln: $(sin(x^2))' = cos(x^2)(x^2)' = 2xcos(x^2)$.
$((x^2+1)^{10})' = 10(x^2+1)^9(x^2+1)' = 10(x^2+1)^9(2x) = 20x(x^2+1)^9$.

9. [Challenge problem, 20 points] Show that $y = sinx$ satisfies the $differential\ equation$ $\frac{d^2y}{dx}+y = 0$. Find a function involving r (where r is any real number) that will satisfy the differential equation $\frac{d^2y}{dx}+r^2y = 0$.

Soln: $\frac{dy}{dx} = (sinx)' = cosx$ and $\frac{d^2y}{dx} = (cosx)' = -sinx = -y$. So we get $\frac{d^2y}{dx} = -y$ and $\frac{d^2y}{dx} + y = 0$. The function $y = sin(rx)$ will satisfy $\frac{d^2y}{dx} + r^2y = 0$ because by chain rule $\frac{dy}{dx} = cos(rx)(rx)' = rcosx$ and $\frac{d^2y}{dx} = (rcos(rx))' = r(cos(rx))' = -r^2sin(rx)$.