Spring 10, Calculus I Quiz 7 Howard University Mathematics
Math 156-05 Sitaraman 4/2/2010

Instructions:
PLEASE PROVIDE STEP BY STEP EXPLANATIONS
Time Limit 30 minutes
Please read the questions carefully before answering
Each problem 5 points unless otherwise stated.

NOTE: THIS QUIZ IS WORTH 40 POINTS (10 POINTS EXTRA CREDIT)

1. (10 points) Graph the curve $y = x^3 - 2x^2 + x + 1$ by finding the points where the function is increasing, decreasing, concave up and concave down. Start by finding the y-intercept and the points where the derivatives are zero. Find all the relative extrema and inflexion points if any.

Soln: The $y-$intercept is $y(0) = 1$. Now $y' = 3x^2 - 4x + 1 = (3x-1)(x-1)$. When $y' = 0$, we have $x = 1/3$ or $x = 1$. There are no points where the derivative is undefined. So 1/3 and 1 are the critical points. Now $y'' = 6x - 4$ and $y'' = 0$ means $x = 2/3$. This is the inflexion point.
When $x = 1/3$ we have $y = 1.15$ and when $x = 1$ we have $y = 1$. When $x < 2/3$ we have $y" = 6x - 4 < 0$ and when $x > 2/3$ we have $y'' > 0$. So the graph is concave down to the left of 2/3 and concave up after 2/3. The derivative $y' = (3x-1)(x-1)$ is positive when $x < 1/3$ and $x > 1$. [You can tell this by plugging in numbers into this product]. It is negative in between 1/3 and 1. So the function is increasing upto 1/3 and decreases from 1/3 rto 1 and then increases again. At $x = 0$ it is 1. So the graph crosses y-axis at 1, increases upto 1/3, then starts bending down. It changes from concave down to concave up at 2/3, and bends upwards again at $x = 1$. It has a relative maximum at $x = 1/3$ and a relative minimum at $x = 1$. This can be seen from both tests: First derivative test says y is increasing before $x = 1/3$ and decreasing after $x = 1/3$. It is decreasing before $x = 1$ and increases after $x = 1$. Second derivative test says at $x = 1$ it is concave down (hence a relative maximum) and at $x = 1/3$ it is concave up and hence a relative minimum.

2. Find the critical points of the function $y = xe^{-x^2}$.

Soln:$y' = x'e^{-x^2}) + x(e^{-x^2}(-2x)) = e^{-x^2}(1-2x^2)$. We have $y' = 0$ when $1 - 2x^2 = 0$ and so $x = \frac{1}{\sqrt{2}}$ and $x = \frac{-1}{\sqrt{2}}$. The derivative $y' = e^{-x^2}(1-2x^2)$ is well defined everywhere, so there are no critical points because of vertical or undefined tangent lines. So $x = \frac{1}{\sqrt{2}}$ and $x = \frac{-1}{\sqrt{2}}$ are the only critical points.

3.(10 points) The population of a town is $P(t) = \frac{100}{1+e^{-t}}$ for $t \geq 0$. By looking at what happens as $t \to \infty$ find the upper limit of the population.Hence find its horizontal asymptote. Also find the critical points and inflexion points.

Soln: $P(t) \to 100$ as $t \to \infty$ because $e^{-t} \to 0$. We have $P'(t) = 100((1+e^{-t})^{-1})' = 100(-1)(1+e^{-t})^{-2}(-e^{-t}) = \frac{100e^{-t}}{(1+e^{-t})^2}$. $P'(t)$ is never zero because numerator $100e^{-t}$ can never be zero. But it is clearly always positive, so the function is always increasing, starting from $P(0) = 100/(1+e^0) = 50$. So the function starts increasing from 50 and keeps increasing and gets closer and closer to 100 though it never reaches it. The graph is asymptotic to $y = 100$. Now the second derivative is, using quotient rule, equal to

$$P''(t) = \frac{100e^{-t}(-1)(1+e^{-t})^2 - 100e^{-t}(2(1+e^{-t})e^{-t}(-1))}{(1+e^{-t})^4}$$

$$\frac{100(1+e^{-t})e^{-t}(e^{-t}-1)}{(1+e^{-t})^4} = \frac{100e^{-t}(e^{-t}-1)}{(1+e^{-t})^3}.$$

This is zero when the numerator is zero, i.e, when $e^{-t} - 1 = 0$ or $t = 0$. It is concave down after $t = 0$ so the function is always concave down.

4.(15 points) Find the absolute maxima or minima in the interval [-2,2] for the function $y = x^3 - x$. Make sure to look at the end-points. Also find the inflexion points. Graph the function and justify the graph using derivatives.

Soln: $y = 0$ when $x^3 - x = 0$. But $x^3 - x = x(x^2-1) = x(x-1)(x+1) = 0$ means $x = 0, 1$ or -1. So the $x-$intercepts are -1, 0 and 1. Now $y' = 3x^2 - 1 = 0$ when $x^2 = 1/3$ which means $x = \frac{1}{\sqrt{3}}$ or $x = \frac{-1}{\sqrt{3}}$.

Now $y'' = 6x$ and it is positive for $\frac{1}{\sqrt{3}}$ and negative for $-\frac{1}{\sqrt{3}}$. So $\frac{1}{\sqrt{3}}$ is a relative minimum and $-\frac{1}{\sqrt{3}}$ is a relative maximum. There are no points where the function or its derivative are not well-defined. To find the absolute maximum or minimum, compare the values at the two relative extrema and the values at the end-points. We get $y = -6$ when $x = -2$, and $y = \frac{2}{3\sqrt{3}}$ when $x = -\frac{1}{\sqrt{3}}$ and $y = 6$ when $x = 2$ and $\frac{2}{3\sqrt{3}}$ when $x = \frac{1}{\sqrt{3}}$. Comparing the four values of y we find that -6 is the absolute minimum and 6 is the absolute maximum in [-2,2]. $y'' = 0$ whenb $x = 0$, so (0,0) is an inflexion point. So the graph would start at -6, increase to $\frac{2}{3\sqrt{3}}$ and then decrease to $\frac{-2}{\sqrt{3}}$ at $\frac{1}{\sqrt{3}}$ and then increase again to 6 at $x = 2$. It will be concave down upto $x = 0$ and concave up after $x = 0$.