11/3/2010 Fall 10, Calculus II Test 3 Sitaraman
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Instructions:
PLEASE PROVIDE STEP BY STEP EXPLANATIONS
ANSWERS WITHOUT EXPLANATION WILL ONLY GET 40 percent
Time Limit 45 minutes
Please read the questions carefully before answering
It is recommended that you try those problems you are most comfortable with, first.
Attempt as many as you can; Anything over 100 is extra credit.

1. (15 pts) Using Simpson's rule with $2n = 10$,[i.e, 10 intervals and 11 points] find the approximate value of $\int_1^{11} ln(x)dx$. Compare with actual value by evaluating the integral.

Soln:
We have $\Delta x = \frac{b-a}{2n} = \frac{11-1}{10} = 1$, $x_0 = 1, x_1 = 2, x_2 = 3, ..., x_9 = 9, x_{10} = 11$.
The approximation is

$$\int_1^{11} ln(x)dx = \frac{1}{3}(\Delta x)[y_0 + 4y_1 + 2y_2 + ...4y_9 + y_{10}]$$

$$= \frac{1}{3}(1)[ln(x_0) + 4ln(x_1) + ... + 4ln(x_9) + ln(x_{10})]$$

$$= \frac{1}{3}[ln(1) + 4ln(2) + 2ln(3) + ... + 4ln(10) + ln(11)] = \frac{49.11319}{3} = 16.3711.$$

Comparing with actual integral (found by integrating by parts, with $du = dx, v = ln(x)$:

$$\int_1^{11} ln(x)dx = \left[xln(x) - \int \frac{dx}{x}\right]_1^{11} = \left[xln(x) - \int \frac{xdx}{x}\right]_1^{11}$$

$$= [xln(x) - x]_1^{11} = [11ln(11) - 11] - [1ln(1) - 1] = 11ln(11) - 10 = 16.3768$$

2. (15 points) Show that the series $\sum_{k=1}^{\infty} \frac{k}{1+k^2}$ is divergent using integral test. You must first show that the function $f(x) = \frac{x}{1+x^2}$ satisfies the relevant criteria.

Soln:
We first show that $f(x)$ is decreasing and continuous on $[1, \infty)$.
Clearly it is a continuous function because it is a rational function (both numerator and denominator are polynomials) whose denominator is never zero in this interval.
To show it is decreasing, look at its derivative, using quotient rule.
We have $f'(x) = \frac{(1)(1+x^2)-x(2x)}{(1+x^2)^2} = \frac{1-x^2}{(1+x^2)^2}$. Clearly the denominator is positive if $x \geq 1$ which is true in $[1, \infty)$.
The numerator is always negative when $x > 1$. So the derivative is negative and function is decreasing in $[1, \infty)$. (note that the function has derivative zero at $x = 1$ and hence neither increasing nor decreasing at $x = 1$, actually a local maximum).

But if $x > 1$ clearly $f(x) < f(1)$ and so the function is decreasing in $[1, \infty)$. So $f(x)$ satisfies criteria for using integral test.

Next we calculate the integral using the substitution $u = 1 + x^2$:

$$\int_1^{\infty} \frac{xdx}{1+x^2} = \lim_{t\to\infty} \int_1^t \frac{xdx}{1+x^2} = \lim_{s\to\infty} \int_2^s \frac{du}{2u}$$

$$= \lim_{s\to\infty} \left[\frac{lnu}{2}\right]_2^s = \lim_{s\to\infty} [(lns/2) - (ln2/2)]$$

[Here $s = 1 + t^2$]. lns goes to infinity as $s \to \infty$, hence the integral diverges, and so given series also diverges.

3. (24 points) Test the following series for convergence. Say what test you are using and show how that test is applied:

$$(a) \sum_{k=1}^{\infty} cos(\pi k) \quad (b) \sum_{k=2}^{\infty} \frac{k}{lnk} \quad (c) \sum_{k=1}^{\infty} k^{-0.9}$$

Soln:

(a) and (b) are divergent by the divergence test. The $k-$th term in both series does not go to zero. Indeed, $cos(\pi k)$ oscillates between -1 and 1 while k/lnk goes to infinity (prove using L'Hospital's rule).

Note: You can also do (a) as a geometric series.

$cos(\pi k) = (-1)^k$ because $cos\pi = -1, cos2\pi = 1, cos3\pi = -1, etc$

This is a geometric series with $r = -1$. Since $|r| = 1$, it diverges.

Also, you can do (b) using comparison with divergent series such as $\sum 1$ or $\sum 1/k$ or even $\sum \sqrt{k}$. The series in (b) can be shown to be bigger than any of those.

(c) is divergent by the $p-$series test because $k^{-0.9} = \frac{1}{k^{0.9}}$ and thus this is a $p-$series with $0.9 = p < 1$ which makes it divergent.

4.(16 pts) Find the limits (i.e, the sums) of the following series:

$$(a) \sum_{k=1}^{\infty} \frac{2^{k+1}}{3^k} \quad (b) \sum_{k=1}^{\infty} \left(\frac{1}{2^k} - \frac{1}{2^{k+1}}\right)$$

Soln: (a) is a geometric series with first term 4/3 and common ratio $a_{k+1}/a_k = 2/3$. So $r = 2/3, a = 4/3$ and the sum is

$$\frac{a}{1-r} = \frac{\frac{4}{3}}{1-\frac{2}{3}} = 4$$

(b) is a telescoping sum and its sum is

$$\lim_{n\to\infty} \left[\left(\frac{1}{2} - \frac{1}{2^2}\right) + \left(\frac{1}{2^2} - \frac{1}{2^3}\right) + ... + \left(\frac{1}{2^{n-1}} - \frac{1}{2^n}\right)\right]$$

$$= \lim_{n\to\infty} \left(\frac{1}{2} - \frac{1}{2^n}\right) = \frac{1}{2}.$$

5. (16 pts) Check for convergence using any test:

$$(a)\sum_{k=1}^{\infty}\frac{2}{k^k} \quad (b)\sum_{k=1}^{\infty}(-1)^k\frac{k}{k+1}$$

Soln:
(a) This is best done using root test. We have

$$\lim_{k\to\infty}\left(\frac{2}{k^k}\right)^{\frac{1}{k}} = \lim_{k\to\infty}\frac{2^{1/k}}{k} = 0$$

In the numerator $2^{1/k} \to 1$ because as $k \to \infty,\ 1/k \to 0$. So the fraction behaves like $1/k$ which goes to 0.

Since the limit is < 1, the series will converge.

This problem can also be done using comparison test. If you ignore the first term, (since removing a finite number of terms does not change convergence), $2/k^k \leq 2/k^2$ and since $\sum 1/k^2$ converges by $p-$series test given series also converges because it is smaller than a convergent series.

(b) This is a diverging alternating series.

We can show that the $k-$th term $(-1)^k a_k = (-1)^k \frac{k}{k+1}$ does not go to 0 as $k \to \infty$.

For k odd, the $(-1)^k a_k$ i.e, 1/2, 3/4, 5/6 ... go to 1 as k goes to infinity and the even terms, i.e, -2/3, -4/5, -6/7 ....go to -1 as k goes to infinity. So $(-1)^k a_k \to 0$ is not possible.

Note that $k/k+1$ is not a decreasing sequence. So you cannot apply the alternating series test.

Also the ratio test for absolute convergence would give the limit as 1 and that is inconclusive.

6. (14 pts) Find the Maclaurin polynomial $p_3(x)$ of degree 3 for $f(x) = e^{2x}$ and estimate the error $R_3(x)$ in the interval [0,1]. Compare with actual error by evaluating $f(0.1) - p_3(0.1)$.

We have

$$p_3(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 = e^0 + 2e^0x + \frac{4e^0}{2!}x^2 + \frac{8e^0}{3!}x^3$$
$$= 1 + 2x + 2x^2 + (4/3)x^3.$$

$R_3(x)$ is estimated by $|R_3(x)| \leq \frac{M}{(4!)}|x|^4$ where M is the maximum value of $f^{(4)}(x)$ in [0,1]. Since e^{2x} is an increasing function, maximum value of $f^{(4)}(x) = 16e^{2x}$ occurs at $x = 1$ and equals $16e^2 = 118.225$. So we get $R_3(0.1) \leq \frac{118.225}{24}(0.1)^4 = 0.0005$ approximately.

Actual error $f(0.1) - p_3(0.1)$ is $e^{2(0.1)} - (1 + 2(0.1) + 2(0.1)^2 + (4/3)(0.1)^3) = 0.00007$ which is well within predicted range.

7. [Challenge problem, 20 points] Let $\sum a_k$ and $\sum b_k$ be two series with positive terms. Show that if $\lim_{k\to\infty}(a_k/b_k) = 0$ and $\sum b_k$ converges, then $\sum a_k$ converges. Apply this with $b_k = k/e^k$ and $a_k = \sqrt{k}/e^k$ to show that $\sum \sqrt{k}/e^k$ converges.

The basic idea is to use the fact that the terms a_k/b_k become very small to show that $\sum a_k$ converges. We do this by writing $a_k = (a_k/b_k)b_k$ and thus the convergence of b_k together with the smallness of a_k/b_k makes the sum of the $a_k's$ converge also.

Let the partial sums $s_n = \sum_{k=1}^{n} a_k$ and $t_n = \sum_{k=1}^{n} b_n$ and $h_k = (a_k/b_k)$. We are given that $h_k \to 0$ as $k \to \infty$ and t_n goes to a finite limit, say T, as $n \to \infty$. Now $s_n = \sum_{k=1}^{n} a_k = \sum_{k=1}^{n}(a_k/b_k)b_k$. Since $h_k \to 0$, we can assume that if there is an N such that if $n > N$ then $h_n < \epsilon$ for any $\epsilon > 0$ however small. Then we can split the sum s_n as follows: $s_n = \sum_{k=1}^{N} a_k + \sum_{k=N+1}^{n} a_k = \sum_{k=1}^{N} a_k + \sum_{k=N+1}^{n} h_k b_k$. Since N is fixed we can say that $s_N = \sum_{k=1}^{N} a_k$ is a fixed number. So $s_n = \sum_{k=1}^{N} a_k = s_N + \sum_{k=N+1}^{n} h_k b_k < s_N + \epsilon(\sum_{k=N+1}^{n} b_k)$ Since $\sum b_k$ converges, the partial sums t_n are also bounded, and so the sum $\sum_{k=N+1}^{n} b_k$ is also bounded. So s_n is also bounded since both terms of the right hand side of the equation for s_n are bounded. Since it is a positive sequence that is increasing $(s_{n+1} = s_n + a_n > s_n)$ and it is bounded, s_n converges as well.