11/17/2010 Fall 2010, Calculus II Quiz 9 solutions Sitaraman

1. Eliminate the parameter and write the equation of the curve in rectangular co-ordinates and also graph the curve: $x = t + 1, \quad y = t^2$.

Soln:

Plugging in $t = x - 1$ we get $y = (x - 1)^2$ which is a parabola with vertex (1,0) and facing up. It will cut the $y-$axis at $y = (0 - 1)^2 = 1$.

2. Find the equation of the tangent line to the curve
$x = \theta + cos\theta, \; y = 1 + sin\theta$ at $\theta = \pi/6$ without eliminating the parameter.

Soln:
$\frac{dy}{dx} = \frac{dy}{d\theta} / \frac{dx}{d\theta} = \frac{cos\theta}{1 - sin\theta}$.
Slope of the tangent line at a point is the derivative at that point.
So the slope of the tangent line at $\theta = \pi/6$ is
$\frac{cos(\pi/6)}{1 - sin(\pi/6)} = \frac{\sqrt{3}/2}{1-(1/2)} = \sqrt{3}$.
The rectangular co-ordinates of this point are obtained by letting $\theta = \pi/6$ in the defining equations $x = \theta + cos\theta, y = 1 + sin\theta$
$(x, y) = (\frac{\pi}{6} + cos(\pi/6), 1 + sin(\pi/6)) = (\frac{\pi}{6} + \frac{\sqrt{3}}{2}, 1 + \frac{1}{2}) = (\frac{\pi + 3\sqrt{3}}{6}, \frac{3}{2})$.
So the equation of the tangent line is $y - \frac{3}{2} = \sqrt{3}(x - (\frac{\pi + 3\sqrt{3}}{6}))$.
After simplifying you get $y = \sqrt{3}(x - \frac{\pi}{6})$.

3. Convert the following curve into rectangular co-ordinates and graph it: $r = cos\theta$.

Soln:

Multiplying both sides by r we get $r^2 = rcos\theta$ which upon using rectangular co-ordinates becomes $x^2 + y^2 = x$ which can be written as $(x^2 - x) + y^2 = 0$.

Completing the square, we get $(x^2 - x + (1/2)^2) + y^2 = (1/2)^2$.
This is same as $(x - \frac{1}{2})^2 + y^2 = \frac{1}{2^2}$.
This is a circle with center (1/2,0) and radius 1/2.

4. (Challenge: 10 points) Show that the tangent at any point (k, α) of the circle $r = k$ is always perpendicular to the radius at that point using the parametric equation $x = kcos\theta, y = ksin\theta$.

$x = kcos\theta,\ y = ksin\theta$ is a parametric equation gives slope of tangent line as $\frac{dy}{dx} = \frac{dy}{d\theta}/\frac{dx}{d\theta} = \frac{kcos\theta}{-ksin\theta} = -cot\theta$. At the point (k, α) we get the slope as $-cot\alpha$.

The radius at (k, α) is along the line $\theta = \alpha$ and its slope is simply $y/x = (rsin\alpha)/(rcos\alpha) = tan\alpha$.

So slope of tangent is -1/(slope of radius). So the two lines are perpendicular.

Since α could be any angle, (k, α) could represent any point on circle.