11/10/2010 Fall 2010, Calculus II Quiz 8 solutions Sitaraman

1. Write the Maclaurin series for $sinhx$ using the definition

$$sinhx = \frac{e^x - e^{-x}}{2}.$$

Say where the series is convergent (find the radius of convergence). [It is enough to find the radius of convergence of e^x and e^{-x} since the given function is a combination of those two].

Soln:

$e^x = 1 + x + \frac{x^2}{2!} + ... + \frac{x^n}{n!} + ...$ This was discussed in class. You can get this using the formula $f(x) = \sum_{n=0}^{\infty} f^n(0)\frac{x^n}{n!}$.

To get the expansion for e^{-x} we can substitute x with $-x$ in the above:

$$e^{-x} = 1+(-x)+\frac{(-x)^2}{2!}+...+\frac{(-x)^n}{n!}+... = 1-x+\frac{x^2}{2!}+....+\frac{(-)^n x^n}{n!}+....$$

Combining the two, we get

$$sinhx = [(1 + x + x^2 + ...\frac{x^n}{n!} + ...) - (1 - x + x^2 - .... + (-1)^n\frac{x^n}{n!} + ...)]/2$$

$$= [2x + 2\frac{x^3}{3!} + ... + 2\frac{x^{2n+1}}{(2n+1)!} + ...]/2 = x + \frac{x^3}{3!} + ... + \frac{x^{2n+1}}{(2n+1)!} + ...$$

This series expansion is valid everywhere because the series for e^x and e^{-x} are convergent everywhere.

NOTE: It is also okay to do this directly by finding derivatives of $sinhx$ to find the coefficients. You get $sinh(0) = 0, sinh'(0) = cosh(0) = 1, sinh''(0) = sinh(0) = 0$, etc., The coefficients alternate between zero and non-zero terms leaving only the odd power terms.

2. Find the radius of convergence and interval of convergence of the power series

$$\sum_{n=0}^{\infty} \frac{x^n}{n+1}$$

Soln: Using the ratio test for absolute convergence we get

$$\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{x^{n+1}}{n+2} / \frac{x^n}{n+1}\right|$$

$$= \lim_{n\to\infty} |x| \frac{n+1}{n+2} = |x| \lim_{n\to\infty} \frac{1+\frac{1}{n}}{1+\frac{2}{n}} = |x|(1) = |x|$$

Thus $\rho = |x|$ and for convergence we need $|x| < 1$. So it will be convergent in $-1 < x < 1$.

Now check at the boundary points 1 and -1.

At $x = 1$ the series becomes $\sum_{n=0}^{\infty} \frac{1}{n+1}$ which is the divergent harmonic series.

At $x = -1$ the series becomes $\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}$ which converges by the alternating series test. So the given series converges in [-1,1) and the radius of convergence is 1.

3. Show using the formula for Maclaurin series that $ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + ...$ You can assume that the series converges to $ln(1+x)$ in the interval $-1 < x \leq 1$. Using this series find the value of $ln(1.5)$ correct to 2 decimal places (error is less than 0.005).

Soln:

We have $f'(x) = \frac{1}{1+x}, f''(x) = \frac{-1}{(1+x)^2}, f'''(x) = \frac{-1(-2)}{(1+x)^3} = \frac{2!}{(1+x)^3}$, and so on and the $n-$th derivative is $f^{(n)}(x) = \frac{(-1)^{n-1}(n-1)!}{(1+x)^n}$.

Plugging in $x = 0$ to find the Maclaurin expansion we get $f(0) = ln1 = 0, f'(0) = 1, f''(0) = -1, ...f^{(n)}(0) = (-1)^{n-1}(n-1)!....$

Plugging these values into the formula $f(x) = \sum_{n=0}^{\infty} f^{(n)}(0)\frac{x^n}{n!}$ we get $ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + ... + (-1)^{n-1}\frac{x^n}{n} + ...$

To find the number of terms needed for approximating to one decimal place we need the error $R_n(x) < 0.05$.

Now by the remainder formula we have $R_n(x) \leq \frac{M|x|^{n+1}}{(n+1)!}$ where M is the maximum of $|f^{(n+1)}(x)|$ in an interval containing $x_0 = 0$.

Taking the interval $[0, 1]$ we see that the maximum of $|f^{(n+1)}(x)| = \frac{n!}{(1+x)^{n+1}}$ is $\frac{n!}{(1+0)^{n+1}} = n!$. [Because $1/(1+x)^n$ is a decreasing function in this interval, its maximum value is assumed at the left endpoint of the interval].

Note: To find $ln(1.5)$ we need to put $x = 0.5$ in $ln(1 + x)$.

Plugging this into the formula for $R_n(0.5)$ we get

$|R_n(0.5)| \leq \frac{n!(0.5)^{n+1}}{(n+1)!} = \frac{(0.5)^{n+1}}{(n+1)}$.

So it is enough if we find n so that $\frac{1}{(0.5)^{n+1}(n+1)} < 0.005$. Using calculator we see that $n = 5$ is the smallest such n.

Note: if you use a different interval containing 0, say (-0.1,1), you may get a different answer. that is okay.

So we find the approximate value as

$ln(1.5) = (0.5) - \frac{0.5^2}{2} + \frac{0.5^3}{3} - \frac{0.5^4}{4} + \frac{0.5^5}{5} = 0.407291667$. Comparing with $ln(1.5) = 0.405465108$ from the calculator we see that the error is 0.00182655889 which is smaller than 0.005.

4. (Challenge: 10 points) Prove using differentiation and other applicable operations on a known series:

$$\sum_{n=1}^{\infty} nx^n = \frac{x}{(1-x)^2} \quad for \ -1 < x < 1.$$

This is problem 36a in 9.10.

We showed in class that $\frac{1}{1-x} = 1 + x + x^2 + x^3 + ... + x^n + ...$ in $-1 < x < 1$.

Since we can differentiate term by term within the interval of convergence, we get $\frac{d}{dx}(\frac{1}{1-x}) = 0 + 1 + 2x + 3x^2 + ... + nx^{n-1} + ...$ as long as $-1 < x < 1$.

So we get (differentiating using chain rule) $\frac{(-1)(-1)}{(1-x)^2} = 1+2x+3x^2+ ... + nx^{n-1} + ...$ in $-1 < x < 1$.

Multiplying both sides by x we get $\frac{x}{(1-x)^2} = x + 2x^2 + 3x^3 + ... + nx^n + ... = \sum_{n=1}^{\infty} nx^n$ in $-1 < x < 1$.