10/27/2010 Fall 2010, Calculus II Quiz 7 Solutions Sitaraman

1. Check if the following series is convergent, and find the limit (the sum) if it does:

$$a)\sum_{n=1}^{\infty}\frac{1}{n^2+n} \qquad b)\sum_{n=1}^{\infty}\frac{1}{\sqrt{n+1}}$$

Soln: (a) This is a telescoping series and it converges. The limit can be found as follows:

$$\sum_{n=1}^{\infty}\frac{1}{n^2+n}=\sum_{n=1}^{\infty}\frac{1}{n(n+1)}=\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right)$$

$$=\lim_{k\to\infty}\left[\left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+....+\left(\frac{1}{k-1}-\frac{1}{k}\right)+\left(\frac{1}{k}-\frac{1}{k+1}\right)\right]$$

$$=\lim_{k\to\infty}[1-\frac{1}{k+1}]=1.$$

[You can also use comparison test because $1/(n^2+n)<1/n^2$ which converges because $p=2$ and $2>1$ but that only shows it is convergent. It doesn't tell you the limit].

(b) This can be checked using the integral test, since $f(x)=\frac{1}{\sqrt{x+1}}$ is a continuous and decreasing function in $[1,\infty)$. We have:

$$\int_1^{\infty}\frac{dx}{x+1}=\lim_{t\to\infty}\int_1^t\frac{dx}{x+1}=\lim_{t\to\infty}\left[\frac{2(x+1)^{3/2}}{3}\right]_1^t$$

We see that the integral is divergent. So the series is also divergent.

[Also you can use comparison with $1/\sqrt{n}$ which is a divergent p series ($p=1/2$). You can show $1/\sqrt{n+1}>2/\sqrt{n}$. Since the given series is bigger than a divergent series, it must diverge as well].

2. Use any method to determine whether the following sequences converge:

$$a)\sum_{n=1}^{\infty}\frac{2^n}{n!} \qquad b)\sum_{n=1}^{\infty}\frac{sin^2n}{n^3}$$

[n is in radians in $sin^2 n$].

Soln: a) The ratio test is best suited to check this series. We have $a_{n+1}/a_n = \frac{2^{n+1}}{(n+1)!}\frac{n!}{2^n} = \frac{2}{n+1}$ which goes to 0 as n goes to ∞. So the series converges.

b) The comparison test is best suited for this. We will compare given series with $\sum_{n=1}^{\infty}\frac{1}{n^3}$. First notice that $0 < sin^2 n < 1$ for all n. Now we have $\frac{sin^2 n}{n^3} < \frac{1}{n^3}$ and we know that $\sum_{n=1}^{\infty}\frac{1}{n^3}$ converges by the $p-$test because $p = 3$ which is bigger than 1. So by the comparison test we get that the given series also converges.

3. Determine whether the following series is convergent and if so, absolutely or conditionally:

$$a)\sum_{k=1}^{\infty}\left(\frac{-1}{4}\right)^k \qquad b)\sum_{k=1}^{\infty}\frac{(-1)^k}{lnk}$$

Soln: a) This is actually a geometic series with $r = -1/4$. Since $|r| < 1$, it is convergent. In fact it is absolutely convergent because the absolute value series is also a geometric series with $r = 1/4$.

b) This converges by the alternating series test because $\frac{1}{lnk}$ goes to 0 and it is decreasing. It can be shown not absolutely convergent by comparison test as follows: The absolute values of the $k-$th term is $1/lnk$ and we have $1/lnk > 1/k$ because $k > lnk$. So by comparison with $\sum_{k=1}^{\infty}\frac{1}{k}$ which is the harmonic series that is known to diverge, we get that the absoulte value series diverges and so the given series is only conditionally convergent.

4. (Challenge: 10 points) Prove using a suitable test that the following series diverges (as always, k is in radians inside the sine function):

$$\sum_{k=1}^{\infty} sin\left(\frac{\pi}{k}\right)$$

This is problem 52, section 9.5 in book. Using limit comparison test with $\sum \pi/k$, we get:

$$\lim_{k\to\infty}\frac{sin(\pi/k)}{\pi/k} = 1$$

because $sinx/x$ goes to 1 as x goes to 0. So given series diverges if $\sum \pi/k$ does. But $\sum \pi/k = \pi(\sum \frac{1}{k})$ which is just π times the harmonic series which is known to be divergent.