1. For the sequence $\frac{3}{4}, \frac{4}{5}, \frac{5}{6}, ...$ write a formula for a_n and find the limit.

Soln: $a_n = \frac{n+2}{n+3}$. To find the limit we can divide both numerator and denominator by the highest power, namely n, to get

$$\lim_{n\to\infty} \frac{n+2}{n+3} = \lim_{n\to\infty} \frac{1+\frac{2}{n}}{1+\frac{3}{n}} = 1.$$

3. Show that the following sequence is decreasing, bounded, and find the lower bound as well as the limit (they could be the same number):

$$a_n = \frac{n}{e^n} \; for \; n = 1, 2, 3, ...$$

Soln: The ratio between successive terms is

$$a_{n+1}/a_n = \frac{\frac{n+1}{e^{n+1}}}{\frac{n}{e^n}} = \frac{n+1}{n}\frac{1}{e^{n+1-n}} = \frac{(1+\frac{1}{n})}{e}$$

Since $e = 2.718$ and $1 + \frac{1}{n} \le 2$ for $n = 1, 2, 3, ...$ we see that the ratio is always smaller than 1, $a_{n+1} < a_n$ for all n and so the sequence is decreasing. [Note that 1,2,3....means "for all natural numbers, starting with 1,2,3" and not just 1, 2 and 3]. Clearly the sequence is always positive, so all the terms in the sequence are bigger than 0 and so 0 is a lower bound. The limit can be found using L'Hospital's rule since both numerator and denominator go to ∞ as $n \to \infty$. We get

$$\lim_{n\to\infty} \frac{n}{e^n} = \lim_{x\to\infty} \frac{x}{e^x} = \lim_{x\to\infty} \frac{x'}{(e^x)'} = \lim_{x\to\infty} \frac{1}{e^x} = 0.$$

3. Determine whether the following series is convergent and if so, find its limit:

$$\sum_{k=0}^{\infty} \left(\frac{3}{4}\right)^{k+1}$$

Soln: This is a geometric sequence with $a = a_0 = 3/4$ and common ratio $r = 3/4$. Here r can be found by dividing a_{n+1} by a_n as follows:

$$a_{n+1}/a_n = \left(\frac{3}{4}\right)^{(n+1)+1} / \left(\frac{3}{4}\right)^{n+1} = \left(\frac{3}{4}\right)^{n+2} / \left(\frac{3}{4}\right)^{n+1}$$

$$= \left(\frac{3}{4}\right)^{n+2-(n+1)} = \left(\frac{3}{4}\right)^{1} = 3/4.$$

You can also say that $\sum_{k=0}^{\infty} \left(\frac{3}{4}\right)^{k+1} = \sum_{k=0}^{\infty} \frac{3}{4} \left(\frac{3}{4}\right)^{k}$ and conclude $a = 3/4, r = 3/4$ because a geometric sequence is of the form $a_n = ar^n$ for $n = 0, 1, 2, 3, ...$

Since $|r| < 1$, the series converges to the limit given by:
$\frac{a}{1-r} = \frac{3/4}{1-\frac{3}{4}} = (3/4)/(1/4) = 3.$

4. (Challenge: 10 points) The famous Fibonacci sequence that is found a lot in nature is given by 1,1,2,3,5,8,13,21,.... Show that this is a recursive sequence and find the relation between a_n, a_{n+1} and a_{n+2}. From this derive also that $\frac{a_{n+2}}{a_{n+1}} = 1 + \frac{a_n}{a_{n+1}}$. Assuming that the limit of the sequence $\frac{a_{n+2}}{a_{n+1}}$ is the same as that ofthe sequence $\frac{a_{n+1}}{a_n}$, show that $L = Lim \frac{a_{n+1}}{a_n} = \frac{1+\sqrt{5}}{2}$. This last number is called the Golden Ratio and it also appears a lot in nature, and is used in architecture as well.

This is a modified version of problem 47, section 9.2 in book. The recursive relation satisfied by the terms of the Fibonacci sequence is $a_{n+2} = a_{n+1} + a_n$. Dividing both sides by a_{n+1} we get the equation $\frac{a_{n+2}}{a_{n+1}} = 1 + \frac{a_n}{a_{n+1}}$. Taking limits of both sides and assuming that both ratios mentioned in the question go to the same limit L we get $L = 1 + \frac{1}{L}$. Simplifying we get $L^2 - L - 1 = 0$. Solving the quadratic equation and taking the positive solution (the solution cannot be negative because the terms of the sequence $\frac{a_{n+1}}{a_n}$ are positive) we get that $L = \frac{1+\sqrt{5}}{2}$.