1. (20 points) Evaluate the following integral using trapezoidal and midpoint methods with $n = 8$ and Simpson's rule with $2n = 8$ [Note that you cannot get the answer for Simpson's method using $S = (2M + T)/3$ with these values of n. In fact $S_8 = [2M_4 + T_4]/3$].

$$\int_3^7 \sqrt{x-2}\, dx$$

Compare answers with actual integral. Rank methods by order of accuracy.

Soln: Trapezoidal method uses the formula

$\left(\frac{b-a}{2n}\right)(y_0 + 2y_1 + ...2y_{n-1} + y_n)$ to approximate the integral. With $b = 7, a = 3, n = 8$, we get that each interval is of length (7-3)/8 = 0.5. So the endpoints are 3,3.5,4,4.5,...,6.5,7.

$\displaystyle\int_3^7 \sqrt{x-2}\, dx = \left(\frac{7-3}{16}\right)(f(3)+2f(3.5)+2f(4)+....2f(6.5)+f(7))$
$= (0.25)(\sqrt{3-2} + 2\sqrt{3.5-2} + ... + 2\sqrt{6.5-2} + \sqrt{7-2}) = 0.25(\sqrt{1} + 2\sqrt{1.5} + ...2\sqrt{4.5} + \sqrt{5}) = 0.25(27.12466) = 6.78116$

For midpoint method we use the formula

$\left(\frac{b-a}{n}\right)(y_{m_1} + y_{m_2} + ...y_{m_{n-1}} + y_{m_n})$ to approximate the integral. The interval length is same (=0.5) but we use the midpoints instead of end-points: $m_1 = \frac{3+3.5}{2} = 3.25$ etc., The approximation is $\displaystyle\int_3^7 \sqrt{x-2}\, dx =$
$\left(\frac{7-3}{8}\right)(f(3.25) + f(3.75) + f(4.25) + ...f(6.25) + f(6.75))$

$= (0.5)(\sqrt{3.25-2} + \sqrt{3.75-2} + ... + \sqrt{6.25-2} + \sqrt{6.75-2}) = 0.5(\sqrt{1.25} + \sqrt{1.75} + ...\sqrt{4.25} + \sqrt{4.75}) = 6.789745817$

For Simpson's method the formula is

$\frac{1}{3}\left(\frac{b-a}{2n}\right)(y_0 + 4y_1 + 2y_2...4y_{2n-1} + y_{2n})$

The approximation is $\displaystyle\int_3^7 \sqrt{x-2}\, dx = \frac{1}{3}\left(\frac{7-3}{8}\right)(f(3)+4f(3.5)+ 2f(4)+....+4f(6.5)+f(7)) = (0.25)(\sqrt{3-2}+4\sqrt{3.5-2}+...+4\sqrt{6.5-2}+ \sqrt{7-2}) = 0.25(\sqrt{1} + 4\sqrt{1.5} + ...4\sqrt{4.5} + \sqrt{5}) = 6.786788333$

The actual integral is

$$\int_3^7 \sqrt{x-2}\, dx = \left[\frac{(x-2)^{\frac{3}{2}}}{3/2}\right]_3^7 = (2/3)[5^{\frac{3}{2}} - 1^{\frac{3}{2}}] = 6.786893258$$

Clearly Simpson's rule gives the more accurate answer. Midpoint method is next (though this will not always be the case!!)

2. (10 points) Evaluate the integrals that converge:

$$(a)\ \int_{-\infty}^0 e^{2x}\, dx \quad (b)\ \int_1^5 \frac{dx}{x-1}$$

Soln:

$$(a)\ \int_{-\infty}^0 e^{2x}\, dx = \lim_{t\to-\infty} \int_t^0 e^{2x}\, dx == \lim_{t\to-\infty} [e^{2x}/2]_t^0$$

$$= \lim_{t\to-\infty} [\frac{1}{2} - \frac{e^{2t}}{2}] = \frac{1}{2}.$$

This is because $e^{2t} \to 0$ as $t \to -\infty$.

$$(b)\ \int_1^5 \frac{dx}{x-1} = \lim_{t\to1^+} \int_t^5 \frac{dx}{x-1} = \lim_{t\to1^+} [ln(x-1)]_t^5 = \lim_{t\to1^+} [ln4 - ln(t-1)]$$

As $x \to 1^+$ we have $x-1 \to 0^+$ and thus $ln(x-1) \to -\infty$. Therefore this limit doesn't exist and so the integral does not converge.

3. (Challenge: 10 points) Write a definite integral that represents the work required to lift a satellite of mass m an "infinite distance" above the earth's surface. Use the fact that the universal law of gravitation says that $F = \frac{GMm}{x^2}$ is the force of gravity between the earth of mass M and the body of mass m and G is the universal constant of gravity. Evaluate this integral in terms of G, M, m and R, the radius of earth.[Note: This result can be used to calculate the escape velocity, i.e, the velocity

required to escape earth's gravity, by equating this work to the kinetic energy of an object launched at velocity v].

This is a modified version of problem 55 in book. The integral is of the form $\int_R^\infty \frac{K dx}{x^2}$ where $F = K/x^2 = \frac{GMm}{x^2}$ is the force of gravity between the earth of mass M and the body of mass m and G is the universal constant of gravity. Upon evaluation, you get $\int_R^\infty \frac{GMm}{x^2} = \lim_{t\to\infty} \int_R^t \frac{GMm}{x^2} = \lim_{t\to\infty} [\frac{-GMm}{x}]_R^\infty = GMm/R$.