9/27/2010 Fall 2010, Calculus II Quiz 4 Solutions Sitaraman

1. Evaluate the following integral: $\int cos^3 x sin^2 x \ dx$

[If you use a relevant identity carefully you should be able to do this with substitution in just one step].

Soln: First write $cos^2 x = 1 - sin^2 x$ Then we get $\int cos^3 x sin^2 x \ dx = \int (cos^2 x cos x sin^2 x) \ dx = \int ((1 - sin^2 x) cos x sin^2 x) \ dx$ This can be done using substitution. Let $u = sinx$. Then $du = cosx$. The given integral becomes $\int ((1 - u^2) u^2) du = \int (u^2 - u^4) du = (u^3/3) - (u^5/5) + C = (sin^3 x/3) - (sin^5 x)/5 + C$.

2. Evaluate $\int sin^4 x \ dx$ by successive reduction.

Soln: The reduction formula (that was derived in class) gives

$\int sin^n x \ dx = \frac{-1}{n} sin^{n-1} x cos x + \frac{n-1}{n} \int sin^{n-2} x \ dx$. Plugging this in we get $\int sin^4 x \ dx = (-1/4) sin^3 x cos x + (3/4) \int sin^2 x \ dx$. Using reduction formula again, we get $\int sin^2 x \ dx = (-1/2) sin x cos x + (1/2) \int sin^0 x \ dx = -(1/2) sin x cos x + (1/2) \int dx = (-1/2) sin x cos x + (x/2)$. Plugging this in for $\int sin^2 x \ dx$ we get

$\int sin^4 x \ dx = (-1/4) sin^3 x cos x + (3/4)[(-1/2) sin x cos x + (x/2)] + C$

$= \frac{-sin^3 x cos x}{4} + \frac{-3 sin x cos x}{8} + \frac{3x}{8} + C$.

NOTE: I didn't take points off if you did it without using reduction formula. One way to do that is by writing $sin^2 x = (1 - cos2x)/2$, then $sin^4 x = (sin^2 x)^2 = (\frac{1 - cos2x}{2})^2$, expanding it, plugging in $cos^2 2x = (1 + cos4x)/2$ and then integrating. Note that integral of $cos2x$ will be $(sin2x)/2$ and integral of $cos4x$ will be $(sin4x)/4$.

3. Evaluate $\int_0^1 \frac{dx}{(1+x^2)^2}$ using trigonometric substitution.

This is of form $a^2 + x^2$ with $a = 1$. We let $x = tan\theta$. Then $1 + x^2 = sec^2\theta,\ dx = sec^2\theta d\theta$. When $x = 0, \theta = 0$ and when $x = 1, \theta = \pi/4$.

Then

$$\int_0^1 \frac{dx}{(1+x^2)^2} = \int_0^{\pi/4} \frac{(sec^2\theta d\theta)}{(sec^2\theta)^2} = \int_0^{\pi/4} \frac{sec^2\theta d\theta}{sec^4\theta}$$

$$= \int_0^{\pi/4} cos^2\theta d\theta == \int_0^{\pi/4} [(1 + cos2\theta)/2] d\theta$$

$$= \frac{1}{2}\int_0^{\pi/4} [1 + cos2\theta] d\theta = \frac{1}{2}\left[\theta + \frac{sin2\theta}{2}\right]_0^{\pi/4} = \frac{1}{2}[\frac{\pi}{4} + \frac{1}{2}] = \frac{\pi}{8} + \frac{1}{4}.$$