12/2/2010 Fall 2010, Calculus II Quiz 11 Solutions Sitaraman

1. Graph the ellipse given by the following equation, marking the foci, vertices and the major and minor axes: $4x^2 + 3y^2 = 12$.

Soln:

Dividing both sides by 12, we get $\frac{x^2}{3} + \frac{y^2}{4} = 1$.

So $a^2 = 4, b^2 = 3$ and $c^2 = a^2 - b^2 = 1$.

So the major axis is along the $y-$axis with $a = 2$, the minor axis $b = \sqrt{3}$ and focus is at a distance of 1 from (0,0).

The vertices are at (0,2) and (0,-2) at a distance of a from (0,0) along y-axis.

The ends of minor axis are at $(0, -\sqrt{3})$ and $(0, \sqrt{3})$.

The foci are at $(0, 1)$ and $(0, -1)$.

2. Graph the hyperbola given by the following equation, marking the foci, vertices and the asymptotes: $x^2 - 2x - 4y^2 = 99$.

Soln:

First we write it as $(x^2 - 2x) - 4y^2 = 99$.

Completing the square on the x part (the $y-$part is already a square) we get $(x - 1)^2 - 1 - 4y^2 = 99$ which gives $(x - 1)^2 - 4y^2 = 100$.

Dividing both sides by 100 we get $\frac{(x-1)^2}{100} - \frac{y^2}{25} = 1$.

So the axis is along the $x-$axis and $a^2 = 100, b^2 = 25$,

$c^2 = a^2 + b^2 = 125$.

Since $a = 10$, and the center is (1,0), the vertices are at $(11, 0)$ and $(-9, 0)$.

The foci are at $(1 + 5\sqrt{5}, 0)$ and $(1 - 5\sqrt{5}, 0)$. [$\sqrt{125} = \sqrt{25(5)} = 5\sqrt{5}$].

The asymptotes are $y = \pm\frac{b}{a}(x - 1)$ which gives $y = \pm(5/10)(x - 1)$.

So the two asymptotes are $y = (1/2)(x - 1)$ and $y = -(1/2)(x - 1)$.

3. From the given equation in polar co-ordinates identify the type of conic and graph the curve marking vertices, foci and directrix (in polar co-ordinates):

$$r = \frac{1}{2 - cos\theta}$$

Soln: Dividing by 2 to get the denominator in the form $1 - ecos\theta$ we get

$$r = \frac{1/2}{1 - (1/2)cos\theta}.$$

Thus $de = 1/2, e = 1/2$. So $e < 1$ and the curve is an ellipse with one focus at the pole to the right of the directrix.

The directrix is at a distance $d = 1$ from the pole.

The verices are obtained by putting $\theta = 0$ and π in the equation.

We get $r = 1$ and $r = 1/3$ respectively. So the vertices are at (1,0) and$(1/3, \pi)$.

Note: When I say $(1/3, \pi)$ it means $r = 1/3, \theta = \pi$. So this vertex is to the left of the focus, at a distance of 1/3.

The focus is at a distance 1/3 from the nearest vertex $(1/3, \pi)$. So by symmetry the other focus is also at a distance of 1/3 from the other vertex and its coordinates are $(2/3, 0)$.

4. (Challenge, 10 points) A whispering gallery is a room in the shape of the top half of an ellipse with major axis along the floor and minor axis perpendicular to the floor. A person standing at one focus can hear something whispered by a person at the other focus. The whispering gallery in the Chicago science museum is 47.3 feet long and the distance from the center of the room to the foci is 20.3 feet. Write the equation of the ellipse in rectangular coordinates with center at (0,0). How high is the room at the center?

Soln:

We are given that $2a = 47.3$ feet and $c = 20.3$ feet. So $a = 23.65$ feet and $b = \sqrt{a^2 - c^2} = \sqrt{23.65^2 - 20.3^2} = 12.1$. But the height of the room at the center is simply the length of the minor axis, that is, b. So height is 12.1 feet.

The equation of the ellipse is

$$\frac{x^2}{23.65^2} + \frac{y^2}{12.1^2} = 1.$$