11/24/2010 Fall 2010, Calculus II Quiz 10 Solutions Sitaraman

1. (20 points) Find the length of arc and area of the dimpled limacon given by $r = 2 + cos\theta$ in the first quadrant. For the length of arc you may use the following: $\int_0^{\pi/2} \sqrt{5 + 4cos(t)}\ dt = 4.29991$ approximately.

Soln:
In the first quadrant θ goes from 0 to $\pi/2$.
The length of arc is

$$\int_0^{\pi/2} \sqrt{r^2 + (dr/d\theta)^2}\ d\theta = \int_0^{\pi/2} \sqrt{(2 + cos\theta)^2 + (-sin\theta)^2}\ d\theta$$

$$= \int_0^{\pi/2} \sqrt{5 + 4cos\theta}\ d\theta = 4.29991$$

The area is given by

$$\int_0^{\pi/2} \frac{r^2}{2} d\theta = \frac{1}{2} \int_0^{\pi/2} (2 + cos\theta)^2\ d\theta$$

$$= \int_0^{\pi/2} 2d\theta + \int_0^{\pi/2} 2cos\theta\ d\theta + \frac{1}{2} \int_0^{\pi/2} cos^2\theta\ d\theta$$

$$= \int_0^{\pi/2} 2d\theta + \int_0^{\pi/2} 2cos\theta\ d\theta + \frac{1}{2} \int_0^{\pi/2} \frac{1 + cos2\theta}{2}\ d\theta$$

Using $\int cos2\theta\ d\theta = (sin2\theta)/2$ we get

$$= \pi + 2 + \frac{1}{4}(\frac{\pi}{2} + 0) = (9\pi/8) + 2 = 5.53429174$$

2. Sketch the parabola, labelling the focus, vertex and the directrix:
$(y - 2)^2 = 3x + 1$

Soln: This opens right as it has a y^2 term with positive coefficient.
Writing in the form $(y - k)^2 = 4p(x - h)$ we get
$(y - 2)^2 = 3(x + \frac{1}{3}) = 4(3/4)(x + \frac{1}{3})$.

So this parabola has vertex $(h, k) = (-\frac{1}{3}, 2)$ and focus is at a distance of 3/4 units from the vertex at $x = -\frac{1}{3} + \frac{3}{4} = \frac{5}{12}$ and directrix is $x = -\frac{1}{3} - \frac{3}{4} = -13/12$.

4. (Challenge, 10 points) A reflecting telescope has a parabolic mirror with a light collector at the focus. If the edge of the mirror has width 10 meters and the depth is 10 meters where is the focus? Write an equation describing the parabola, with its vertex at (0,0) and axis of symmetry along $y-$axis, facing up.

Soln:

The equation of the parabola can be written as $4py = x^2$ where p is the distance of the focus from the vertex. If the width of the edge is 10 meters, then on one side of $y-$axis it extends 5 meters and the height here is 10 meters. Putting $x = 5, y = 10$ in the above equation we get $5^2 = 4p(10) \Rightarrow p = \frac{25}{4(10)} = 0.625$ meters or 62.5 cms. So the focus should be placed 0.625 meters or 62.5 centimeters above the vertex, or at (0, 0.625).