Fall 10, College Algebra I Test 3 Solutions Howard University Mathematics
Math 006-16 Sitaraman 11/3/2010

1.Solve the inequality $|x-3|<5$ and write the solution set in interval form.

Soln:

$$|x-3|<5 \Rightarrow x-3<5 \; or \; x-3>-5$$

This really means $x-3$ is *between* -5 and 5. So we get

$$-5<x-3<5 \Rightarrow -2<x<8$$

In interval form the solution set is (-2,8)

2. Alex invests $12,570 in stocks and bonds. Stocks yield 8 percent return and bonds yield 4 percent return. If total return is $644, how much did Alex invest in each?

Soln:
Let x be the amount invested in stocks.
Then $12,570-x$ is the amount invested in bonds.
Total return is return from stocks plus return from bonds.
Return from stocks is 0.08 times x (8 percent = 0.08).
Similarly return from bonds is 0.04 times $12,570-x$.
Therefore we get that the total return is

$$644=0.08x+0.04(12,570-x).$$

Simplifying and keeping x on one side we get

$$644-(0.04)(12,570)=0.08x-0.04x=0.04x.$$

$$Therefore \quad x=(644-(0.04*12,570))/0.04=3530$$

$$and \quad 12,570-x=9040.$$

Thus 3530 was invested in stocks and 9040 in bonds.
Check: $3530(.08)+9040(0.04)=644$.

3. (20 points) Graph the curve represented by $y-x^3+x=0$. You must show at least three points, none of which is an intercept. Find the intercepts using this equation and also test for symmetry about the x and y axes as well as (0,0).

Soln:
The graph is in file titled "test3 graphs" that is on the update page immediate below "test 3 solutions."

You can get the coordinates of points easily by taking various values of x and plugging into equation. Thus when $x=2, y-x^3+x=y-2^3+2=0$ gives $y=6$. Similarly we get (-2,6), (3,24) and so on.

To get intercepts:

When $x=0$ we get $y=0$. So (0,0) is both an x and y intercept because both x *and* y are 0.

When $y = 0$ we get $-x^3 + x = 0$ which means $x(-x^2 + 1) = 0$.
But $-x^2 + 1 = 1 - x^2 = (1 - x)(1 + x)$.
When this is 0 we get $x = 1$ or $x = -1$.
So the $x-$intercepts are 1 and -1.
To check for symmetry about $x-$axis:
we replace y with $-y$ and check if equation changes.
We get $-y - x^3 + x = 0$ which is not the same equation
(even if you multiply all by -1 you get $y + x^3 - x = 0$
which is also not the same equation).
About the $y-$axis: $y - (-x)^3 + (-x) = y - (-x^3) - x = y + x^3 - x = 0$.
This is also not the same equation.
About the origin: $-y - (-x)^3 + -x = 0 \Rightarrow -y - (-x^3) - x = 0 \Rightarrow -y + x^3 - x = 0$
This is actually same as $y - x^3 + x = 0$. (multiply by -1 on both sides).
So the graph is not symmetric about the two axes but it is symmetric about (0,0).

4.(20 points) Given that the diameter of a circle has endpoints (-1,3) and (4,6) find the center and radius of the circle using the midpoint and distance formulae respectively. Write the equation of this circle in standard form and then the general form.

Soln: The center is the midpoint of the diameter. Midpoint formula gives center as $(\frac{-1+4}{2}, \frac{3+6}{2}) = (1.5, 4.5)$.

The radius is the distance from the center to any one of the end points. Picking (-1,3), we get $r = \sqrt{(-1 - 1.5)^2 + (3 - 4.5)^2} = \sqrt{(-2.5)^2 + (-1.5)^2} = \sqrt{6.25 + 2.25} = \sqrt{8.5}$. From this we get $r^2 = 8.5$.

The standard form of equaion of circle is $(x - h)^2 + (y - k)^2 = r^2$. Plugging in the values $(h, k) = (1.5, 4.5)$ and $r^2 = 8.5$ we get $(x - 1.5)^2 + (y - 4.5)^2 = 8.5$.

Expanding the standard form equation we get the equation in general form as
$x^2 - 2(1.5)x + 1.5^2 + y^2 - 2(4.5)y + 4.5^2 = 8.5$
$\Rightarrow x^2 - 3x + y^2 - 9y + 14.5 = 0$.

5. Find the center and radius of the circle $x^2 + 8x + y^2 + 6y = 0$. Graph the circle marking the center and radius.

Soln:
First we collect x and y terms together.
$(x^2 + 8x) + (y^2 + 6y) = 0$.
Completing the square (adding the square of half the $x-$coefficient to *both* sides)
we get$(x^2 + 8x + 4^2) + (y^2 + 6y + 3^2) = 4^2 + 3^2 = 5^2$.
This is same as $(x + 4)^2 + (y + 3)^2 = 5^2$.
Comparing with $(x - h)^2 + (y - k)^2 = r^2$ we get
$x - h = x + 4$ and $y - h = y + 3$. Also $r^2 = 5^2$. Thus the center is (-4,-3) and radius is 5.

Graph is in "test 3 graph " file.

6. Write the equation of line perpendicular to $x = 2$ and passing through $(3, 10)$.

Soln:
$x = 2$ is a vertical line, so the line perpendicular to it will be horizontal.
It will be of the form $y = k$.

[You can get this from $y = mx + k$ by letting $m = 0$. Here $m = 0$ because slope is 0].

Since it passes through (3,10) we put $y = 10$ and get $k = 10$. So the desired equation is $y = 10$.

7. Write the equation of line passing through (-3,-4) and (-11.2, 12.4).

Soln:
Slope of this line is $m = \frac{12.4 - -4}{-11.2 - -3} = \frac{16.4}{-8.2} = -2$.
So putting $m = -2$ in $y = mx + b$ we get that the equation is of the form $y = -2x + b$.
Plugging in (-3,-4) we get $-4 = -2(-3) + b = 6 + b$ which gives $b = -10$.
So the equation is $y = -2x - 10$.
Check answer by plugging in (-11.2,12.4) : 12.4 = -2(-11.2)-10

8. The pressure of a gas varies inversely as the volume and directly as the temperature. Write the equation relating them. If the pressure is 10 Pascals when the volume is 3 cubic meters and temperature is 100 Kelvin, find the pressure when volume is 2 cubic meters and temperature is 200 Kelvin.

Soln:
The equation is $P = \frac{KT}{V}$.
When $P = 10, T = 100, V = 3$. We get $10 = K(100)/3$. So $K = 30/100 = 0.3$.
So the equation is $P = \frac{0.3T}{V}$.
When $T = 200, V = 2$ we get $P = 0.3(200)/2 = 30$ Pascals.

9. [Challenge problem, 20 points extra credit] Find the intersection points of the line $y = mx$ and the circle $x^2 + y^2 = 1$. Show that the only values of m when the intersection points have integers as co-ordinates is when $m = 0$ and the intersection points are (1,0) and (-1,0). [It is not enough to plug in (1,0) and (-1,0) and show they are intersection points].

Soln: Intersection points can be obtained by putting mx for y in the equation of the circle. We get $x^2 + (mx)^2 = 1$ which means $x^2 + m^2x^2 = 1$ which gives $x^2 = 1/(1 + m^2)$. So $x = \pm 1/\sqrt{1 + m^2}$ and $y = \pm m/\sqrt{1 + m^2}$. The two intersection points are therefore $(\frac{1}{\sqrt{1+m^2}}, \frac{m}{\sqrt{1+m^2}})$ and $(-\frac{1}{\sqrt{1+m^2}}, -\frac{m}{\sqrt{1+m^2}})$.

If the coordinates of the intersections are integers that means $\pm 1/\sqrt{1 + m^2} = n$ where n is an integer. This means $\sqrt{1 + m^2} = n$ or $m^2 + 1 = n^2$ and the only time this happens is for $m = 0, n = 1$.

Proof: $[m^2 + 1 = n^2 \Rightarrow m^2 - n^2 = -1 \Rightarrow (m - n)(m + n) = -1$. This is only possible if $m - n = -1, m + n = 1$ or $m - n = 1, m + n = -1$, because both $m - n$ and $m + n$ would have to divide -1. In both cases you get $m = 0$. So the intersection points are (1,0) and (-1,0)].