11/17/2010 Fall 10, College Algebra I Quiz 9 Sitaraman
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Instructions:
PLEASE PROVIDE STEP BY STEP EXPLANATIONS
WRITING ONLY ANSWERS WILL NOT GET FULL CREDIT
Time Limit 30 minutes
Please read the questions carefully before answering
Each problem 5 points unless otherwise stated.
Any points you get in excess of 30 is extra credit.

1. (10 points) Graph the parabola $y = x^2 - 2x$. Is it facing up or down? What is its vertex and what is its axis of symmetry? What are the x and y intercepts?

Soln:

This is a parabola facing up because x^2 coefficient is 1 and hence positive.

Completing the square on the RHS we get
$y + (-1)^2 = x^2 - 2x + (-1)^2 = (x - 1)^2$.

So the equation is $y + 1 = (x - 1)^2$ which gives $y = (x - 1)^2 - 1$.

Looking at the $(x - 1)^2$ term on RHS we find that the axis of symmetry is $x = 1$.

[You can also get this by saying $x = -b/(2a) = -(-2)/2 = 1$].

When $x = 1$, $y = 1^2 - 2(1) = -1$ so the vertex is at (1,-1).

[You could also say that the graph of $y = x^2$ is moved by 1 to the right when x^2 is replaced by $(x-1)^2$ and then moved down by 1 because you have $(x - 1)^2 - 1$.].

When $x = 0$ we get $y = 0$. So (0,0) is both an x and y intercept.

When $y = 0$ we get $x^2 - 2x = 0$ which gives $(x - 2)x = 0$ which means $x = 2$ or $x = 0$.

So (2,0) is another x intercept.

2. (10 points) Given that the quantity sold (x) of a certain product is a linear function of the price as given by $x = f(p) = -20p + 500, \quad 0 \leq p \leq 25$, express revenue as a function of p say $R(p)$. What is the domain and range of $f(p)$? What price p maximizes revenue?

Soln:

Domain of $f(p)$ is [0,25]. Range is [0,500]. This can be seen by graphing the line $x = -20p + 500$.

Revenue is price times quantity sold. Here price is p and quantity sold is x. So we get

$R(p) = px = p(-20p + 500) = -20p^2 + 500p$.

So the graph of the revenue function is a parabola facing down (because coefficient of p^2 is negative). This means it has a maximum. The maximum is attained at the vertex which is given by $p = -b/2a = -500/(2(-20)) = -500/-40 = 12.50$ dollars.

3. (10 points) Solve the inequality $x^2 - 2x < 4$.

Soln: First we convert this to $x^2 - 2x - 4 < 0$. To see where $x^2 - 2x - 4$ is negative we need to understand the graph of $y = x^2 - 2x - 4$.

METHOD 1:

Because x^2 has positive coefficient, we get that the graph is facing up.

The axis of symmetry is at $x = -b/2a = -(-2)/(2(1)) = 2/2 = 1$.

When $x = 1$ we get $y = 1^2 - 2(1) - 4 = -5$. So the vertex is at $(1, -5)$.

The $x-$intercepts are given by letting $y = 0:$

$0 = x^2 - 2x - 4$ means $x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(-4)}}{2} = 1 \pm \frac{\sqrt{20}}{2} = 1 \pm (\sqrt{4}\sqrt{5}/2) = 1 \pm \sqrt{5}$.

So the $x-$intercepts are $1 + \sqrt{5}$ and $1 - \sqrt{5}$.

The $y-$intercept is -4 [Plug in $x = 0$.]

The parabola starts at (1,-5) and goes up, cutting $y-$axis at -4 and passing through the $x-$axis at $1 + \sqrt{5}$ and $1 - \sqrt{5}$.

So it is negative between the two $x-$intercepts and so the solution set is $1 - \sqrt{5} < x < 1 + \sqrt{5}$.

In interval form it is $(1 - \sqrt{5}, 1 + \sqrt{5})$.

NOTE: You can do this without finding the vertex, graphing the parabola etc., using *only* the x intercepts. Check one number to the left of $1 - \sqrt{5}$, one number between $1 - \sqrt{5}$ and $1 + \sqrt{5}$ and one number to the right of $1 + \sqrt{5}$. For example, $x = 0$ is between the two intercepts. Plugging in 0, we get $0^2 - 2(0) - 4 = -4$ which is negative. So the function must be negative between the two x intercepts.

METHOD 2:

Using completing the square: Add square of half of $x-$coefficient to both sides.

$y + (-1)^2 = (x^2 - 2x + (-1)^2) - 4$ gives

$y + 1 = (x - 1)^2 - 4$ which is same as $y = (x - 1)^2 - 5$.

Looking at $(x - 1)^2$ we see that axis of symmetry is at $x = 1$.

Putting $x = 1$ we get $y = (1 - 1)^2 - 5 = 0 - 5 = -5$.

So vertex is (1,-5).

Putting $y = 0$ we get $0 = (x - 1)^2 - 5$ which means $(x - 1)^2 = 5$.

Solving, we get $x = 1 \pm \sqrt{5}$.

The rest is as in method 1.

4.(bonus 10 points) A parabolic arch has a span of 120 feet and a maximum height of 25 feet. Choosing a suitable coordinate for the vertex, write an equation for the parabola (your equation will depend on the co-ordinates you choose). Calculate the height of the arch at a point 10 feet from the center.

We can choose (0,25) for the vertex.

Then we need a parabola facing down, with y axis as axis of symmetry and its vertex is moved up by 25. Its center will then be at (0,0).

$y = -kx^2$ is the equation of a parabola facing down, symmetric about $y-$axis, for any positive real number k.

So $y = -kx^2 + 25$ has vertex at (0,25).

We also need the $x-$intercepts to be at 60 and -60 (total span is 120 feet).

So when $y = 0$, $x = 60$ or $x = -60$.

But when $y = 0$, we have $-kx^2 + 25 = 0$ or $x = \pm\sqrt{25/k} = \pm 5/\sqrt{k}$.

So we need $5/\sqrt{k} = 60$ which gives $\sqrt{k} = 5/(60) = 1/(12)$.

Squaring, we get $k = 1/(12)^2 = 1/144$.

So $y = -\frac{x^2}{144} + 25$ is the equation of desired parabola.

To find the height 10 feet from center, plug in $x = 10$.

You get $y = \frac{-10^2}{144} + 25 = (-100/144) + 25 = 3500/144 = 24.3$

NOTE: You can also do this as follows.

The equation of a parabola with vertex$(h, k) = (0, 25)$ is $y = a(x - h)^2 + k = a(x - 0)^2 + 25 = ax^2 + 25$.

Now as mentioned above, $x = 60$ when $y = 0$ according to the coordinates we have selected.

Plugging in these values, we get $0 = a(60)^2 + 25$.

Solving for a, you get $a = -25/(60^2) = -1/144$ resulting in the equation $y = -\frac{x^2}{144} + 25$.