12/2/2010 Fall 10, College Algebra I Quiz 11Soluions Sitaraman
Math 006-16 Howard University Mathematics Dept

1. (15 points) The population of a state is given by $P(t) = 2500e^{0.06t}$, where t is measured as the number of years after 1900. [So $t = 0$ in 1900, $t = 10$ in 1910, $t = 100$ in 2000, etc.,].

What was the population at the beginning, i.e, 1900? At what rate is the population growing? What will its population be in 2050?

Soln:

Population in 1900 is $P(0) = 2500e^{0.06(0)} = 2500$.

Growth rate is 0.06 or 6 percent.

In 2050 we will have $t = 150$. So the population will be $P(150) = 2500e^{0.06(150)} = 2500e^9 = 20257709.8$ or after rounding 20,257,710 –that is 20 million, 257 thousand and 710.

2. For $f(x) = (x^2 - 9)(x^2 - 16)$ answer the following:

(3 points) What is the degree?

(4 points) List the roots and their multiplicities. At each root say whether graph crosses or touches $x-$axis .

(2 points) What is the $y-$intercept?

(6 points) Graph the curve. You must label all intercepts.

Soln:

Highest degree term is obtained from multiplying the x^2 terms together. You get x^4. So degree is 4. [If you want, check by multiplying out the two terms together. You will get a polynomial that starts with x^4. In fact, it is $x^4 - 25x^2 + 144$].

The roots are obtained by factoring:

$(x^2 - 9)(x^2 - 16) = (x - 3)(x + 3)(x - 4)(x + 4)$. The roots are 3,-3,4,-4. All of them occur with multiplicity 1 because the powers of $x + 3, x - 3, x + 4, x - 4$ are all 1 in the polynomial's factorization. So the graph crosses the x-axis at all these roots.

y-intercept is obtained by letting $x = 0$. So $f(0) = (-9)(-16) = 144$.

Graph is on update page – click on "quiz11graph"

3.(bonus 10 points) In problem 1 compare the population after t years with the population after $t + 1$ years. For example, write the change in population from 1910 to 1911 as a fraction of the population in 1910.

Prove that you will get the same result in general by deriving the following formula for $P(t) = Ae^{kt}$:

$$\frac{P(t+1) - P(t)}{P(t)} = k \textit{ approximately}$$

[You may use the fact that $e^k - 1$ is approximately k when k is small].

The basic idea is that the growth rate is the same at any time, and that the percentage increase is given by k in the formula $P(t) = Ae^{kt}$. [In problem 1, for example, $k = 0.06$].

$P(t+1) = Ae^{k(t+1)} = Ae^{kt+k} = Ae^{kt}e^k = P(t)e^k$.

So when t is increased by 1 the population is multiplied by e^k.

In 1910, $t = 10$.

$$\frac{P(11) - P(10)}{P(10)} = \frac{2500e^{0.06(11)} - 2500e^{0.06(10)}}{2500e^{0.06(10)}}$$
$$= \frac{e^{0.66} - e^{0.6}}{e^{0.6}} = 0.0618365465$$

which is very close to 0.06 itself.

In general:

$$\frac{P(t+1) - P(t)}{P(t)} = \frac{(P(t)e^k) - P(t)}{P(t)} = e^k - 1$$

which is approximately equal to k when k is small.

Note that this is irrespective of t!!

FYI: Why is $e^k - 1$ close to k? Recall that when growth is continuous, we replace $(1+\frac{k}{n})^n$ by e^k in the compund interest formula and elsewhere. Now when k is small, you can check using calculator that $(1+\frac{k}{n})^n$ is close to $1+k$. See table 8 on page 469 in book. The bigger the n, the closer they are. For example, $(1+\frac{k}{2})^2 = 1 + (k/2)^2 + 2(1)(k/2) = 1 + k + k^2$. If $k = 0.06, n = 2$ then $k^2 = 0.0036$ and $1 + k + k^2 = 1.0636$ whereas $1 + k = 1.06$. They get closer as k gets smaller and n increases. So e^k is gets close to $(1+\frac{k}{n})^n$ which gets close to $1+k$. So $e^k - 1$ gets close to k.