Fall 09, Test 3, Probability and Statistics I Math 189-1 Sitaraman
11/18/09 Howard University Mathematics

Instructions:
PLEASE PROVIDE STEP BY STEP EXPLANATIONS
Time Limit 45 minutes
Please read the questions carefully before answering
Each problem 10 points unless otherwise stated.

1. The geometric distribution has pdf $f(x) = p(1-p)^{x-1}, x = 1, 2, 3, \ldots$ where $f(x)$ is the probability of the first success happening in the $x-$th Bernoulli trial with probability of success p.
Show that $M(t) = \frac{pe^t}{1-((1-p)e^t)}$ and $\mu = 1/p$.

Soln: $M(t) = \sum_{x=1}^{\infty} e^{tx} p(1-p)^{x-1} = pe^t[\sum_{x=0}^{\infty}(1-p)^x e^{tx}]$. Notice that the series invoilved here is a geometric series. Using the formula $S = \frac{a}{1-r}$ for the sum of a geometric series with first term a and common ration r, we get $M(t) = \frac{pe^t}{1-((1-p)e^t)}$. Differentiating, we get $M(0) = 1/p$.

2. (20 points) Telephone calls arrive at the rate of 2 per minute at a booth in a Poisson process. Let X denote the number of calls in a 2 minute period. What is the probability of getting 3 or 4 calls?

Soln: The average per unit time is 2. So for 2 minutes the average is $\lambda = 4$. Now $P(3 \leq X \leq 4) = P(X \leq 4) - P(X \leq 2) = 0.629 - 0.238 = 0.391$ or 39.1 percent using the tables.

3. The number of patients arriving at a health center each half hour are

3 3 2 0 4 5 6 4 4 3 2 1 2 3 0

5 5 3 2 3 5 4 1 2 0 3 2 4 2 6

Find the sample mean and variance. Comparing the mean and variance, do you think that this could be the observations of a Poisson random variable?

Answer: The sample mean is 2.97. The sample variance is $1.67^2 = 2.79$. Since they are close to each other, this could be the observations of a Poisson random variable.

4. Construct a stem and leaf table (with integer stems) for the following data and find the median, mode and the first and third quartile:

11 24 15 32 29 17 31 37 24 19 9 10 20 35 41 54 12 16 24 17

Soln: Arranging in ascending order, we see that the median is the average of 20 and 24, namely 22. The mode is 24 because it appears 3 times. The first quartile (5th entry out of 20) is 15 and the third quartile is 31.

5. Show that the pdf $f(x)$ of the geometric distribution from problem 1 is a valid distribution. If x were assumed to be a continuous variable taking values from 1 to ∞, then show that $e^p = \frac{1}{1-p}$ must be true. (Challenge, extra credit 10 points: Using derivatives, show that this is only true if $p = 0$).

Soln: We need to show that $\sum_{x=1}^{\infty} p(1-p)^{x-1} = p[\sum_{x=0}^{\infty}(1-p)^x] = \frac{p}{1-(1-p)} = 1$. [Note that we could have proved this also by calculating $M(0)$]. If X were continuous, we need to replace the summation with an integration, getting $\int_1^{\infty} p(1-p)^{x-1}dx = 1$. Upon integration, we get $\left[\frac{e^{ln((1-p)x)}}{ln(1-p)}\right]_1^{t\to\infty} = 1$. Now, if $0 < p < 1$, we have $0 < 1-p < 1$ as well, and $ln(1-p) < 0$ and $\frac{e^{ln((1-p)x)}}{ln(1-p)} \to 0$ as $x \to \infty$. So we get $\frac{p}{1-p}\left(-\frac{e^{ln(1-p)}}{ln(1-p)}\right) = 1$ from which we get the desired answer. To answer the challenge, note that $f(x) = (1-x)e^x - 1$ has $f(0) = 0, f'(0) = -xe^x|_{x=0} = 0$, and $f"(0) = -xe^x - e^x|_{x=0} = -1$. So clearly 0 is a local maximum in [-1,1]. Now clearly $f'(x) < 0$ for all x in (0,1) and hence $f(x)$ is a decreasing function in (0,1). Therefore $f(0) = 0$ has to be a global maximum in [0,1). Because if there were a value of $f(x)$ in (0,1) bigger than 0 then using mean value theorem we could show that the derivative of f will have to be positive for some value in (0,1) as well. [Note: We are restricting ourselves to (0,1) because p lies in that interval]. This means $f(x)$ is negative in (0,1). Since $1-x > 0$ in (0,1), $f(x)/(1-x) < 0$ in (0,1) as well. Thus $e^x - \frac{1}{1-x} < 0$ in (0,1). Since $e^0 = 1/(1-0) = 1$, we get that 0 is the only value in (0,1) for which $e^x = \frac{1}{1-x}$.

6.(20 points) Customers arive at a shop according to a Poisson process with an average of 3 per hour. Let X denote the waiting time until the first customer arrives. Find $P(X > 0.5)$ and the probability of waiting for another 15 minutes given that no customer arrived in 30 minutes (0.5 hours).

Soln: X has an exponential distribution with $\theta = 1/\lambda = 1/3$ and pdf $f(x) = \frac{e^{-x/(1/3)}}{(1/3)} = 3e^{-3x}$. Now $P(X > 0.5) = \int_{0.5}^{\infty} 3e^{-3x}dx = e^{-3x}|_{x=0.5} = e^{-1.5} = 0.2231$ or 22.31 percent. Also, $P(X > 0.75 \mid X > 0.5) = \frac{P(X>0.75)}{P(X>0.5)} = \frac{e^{-3(0.75)}}{e^{-3(0.5)}} = e^{-3(0.25)} = e^{-0.75} = 0.4724$ or 47.24 percent. Note that this is really just the probability of waiting more than 15 minutes.

7. If the number of points scored in a basketball game follows a Poisson process with average of 2 per minute, find the probability of waiting less than 3 minutes until the first two points are scored?

Soln: We have $\lambda = 2, \theta = 0.5$. The probability of waiting for the first two points follows a Gamma distribution with $\alpha = 2$. We get $F(3) = \int_0^3 \frac{x^{2-1}e^{-x/(0.5)}}{\Gamma(2)(0.5)^2}dx = 4\int_0^3 xe^{-2x}dx = 0.2457$ or 24.57 percent.

8. Find the mean of the exponential distribution that has the same pdf as $\chi^2(2)$.

Soln: If θ is the mean, then we have that $\frac{e^{-x/\theta}}{\theta} = \frac{x^{\frac{2}{2}-1}e^{-x/2}}{\Gamma(2/2)2^{2/2}} = \frac{e^{-x/2}}{2}$. This is true if $\theta = 2$.