Howard University Math Department

PLEASE STUDY CLASS NOTES, HW PROBLEMS AND PRACTICE PROBLEMS IN ADDITION TO THESE.

1. True or False ? Prove if true and provide counterexample or disprove if false.
   (a) A unit in a quadratic ring $\mathbb{Z}[\sqrt{d}]$ is always of norm 1.
   (b) Every Gaussian integer has a square root that is also a Gaussian integer
   (c) The equation $x^2 - 7y^2 = -1$ has no solutions in integers. (what about in real numbers?)
   (d) The prime 11 is also a Gaussian prime.
   (e) $3 + i$ is an irreducible (hence prime) Gaussian integer.
   (f) For any $n \in \mathbb{Z}$, $n^2 + 1$ is either odd or 2 times an odd number.
   (g) The negative Pell's equation $x^2 - 20y^2 = -1$ has a solution.
   (h) If there is no non-trivial solution for $x^p + 9y^p = z^p$ then there is no non-trivial solution for $x^{pk} + 9y^{pk} = z^{pk}$ for any k.
   (i) If $m^2 + n^2$ is divisible by an odd prime p then $p \equiv 1 \pmod 4$.
2. Prove using properties of a cyclic group of order n that $\sum_{d|n} \phi(d) = n$.
3. If p divides $2^k - 1$ but not for any $n < k$ then show that k divides $p - 1$. In fact show that k divides m for any m for which $2^m \equiv 1 \pmod p$.
4. Find a fundamental solution and the first three solutions of $x^2 - 7y^2 = 1$.
5. Describe the group of units of $\mathbb{Z}[\sqrt{7}]$. (Look at 1c before you answer).
6. Prove that there are infinitely many square triangular numbers and explain how to produce them.
7. Describe all the units of $\mathbb{Z}[\sqrt{3}i]$ (Use norm multiplicativity).
8. Find the greatest common divisor of $5 + 3i$ and $5 + i$. Also find the full factorization into Gaussian primes of the two numbers.
9. (Easy if you get the trick) Show that $x^2 - xy + y^2 = 1$ has only the following solutions in integers, namely $(\pm 1, 0), (0, \pm 1), (1, 1), (-1, -1)$. Note that LHS is the norm of $x + y\rho$ in $\mathbb{Z}[\rho]$ where $\rho = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$. So this is saying that the only units in that ring are $\pm 1, \pm\rho, \pm(1 + \rho)$.

10. Give an example of a negative Pell's equation $x^2 - Dy^2 = -1$ with a fundamental solution. Use it to find two more solutions of that equation as well as three solutions of $x^2 - Dy^2 = 1$.

11. Show that the square root of $3 + 2\sqrt{2}$ is also in $\mathbb{Z}[\sqrt{2}]$ and that the square root of $2i$ is also a Gaussian integer.

12. Find the number of ways to write 2925 as a sum of two squares.

13. This is related to the ring of algebraic integers A of $K = \mathbb{Q}[\sqrt{D}]$.

    Reference: K. Conrad's notes on quadratic extensions "Factoring in quadratic fields."

    a) Show that $\mathbb{Z}[\sqrt{D}] \subseteq A$ always and $\mathbb{Z}[\frac{1+\sqrt{D}}{2}] \subseteq A$ if $D \equiv 1 \pmod 4$.

    b) Show that $\mathbb{Z}[\sqrt{D}] = A$ if $D \equiv 3 \pmod 4$ and $\mathbb{Z}[\frac{1+\sqrt{D}}{2}] = A$ if $D \equiv 1 \pmod 4$.

14. This concerns the equation $10^x - y^2 = n$ where x, y, n are positive integers.

    Reference: "Elementary Proof of the Completeness of OEIS A051221 Below 2000" by Seiichi Azuma.

    Show that this can be converted to a Pell type equation (hint: let $x = 2u + 1$).

15. Show that any prime dividing the Fermat number F_n is 1 modulo 4.

16. Show that the Euler function ϕ is multiplicative.

17. If the prime $q \equiv 1 \pmod 4$ and $p = 2q + 1$ is also a prime, show that 2 is a primitive root mod p.

18. Let p be an odd prime and g a primitive root mod p.

    (a) Show that g^k is a quadratic residue mod p iff k is even.

    (b) Prove using (a) Euler's criterion:

$$a^{\frac{p-1}{2}} = \left(\frac{a}{p}\right) \pmod p.$$

19. If (x, y, z) is a solution for $x^4 + y^4 = z^2$ then show that we can use them to produce a solution for $X^2 + 4Y^4 = Z^4$.

20. Compute $\phi(12)$ using only the values of $\phi(1), \phi(2), \phi(3), \phi(4)$ and $\phi(6)$. Then verify your answer using direct computation.

21. Is it possible to write 1885 as a sum of two squares? As the hypotenuse of a primitive pythagorean triple?

22. Using the fact that 13 divides $26 = 5^2 + 1$ find a, b such that $13 = a^2 + b^2$.

23. Show that there is only one way to write $p = a^2 + b^2$ with $a \geq b \geq 0$.