10-3-2025 Notes, Proofs and Problem Solving 1

Proof by Strong Induction

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Outline

Proof by Induction: Strong version

1. First prove statement for $n = 1, 2, 3, ..m$ where m is chosen suitably.
2. Next assume it true for $k < n$
3. Using the previous case, prove for n.

Definition

Fibonacci number F_n is defined by the *recurrence relation*

$$F_n = F_{n-1} + F_{n-2},\ F_1 = 1, F_2 = 1.$$

So $F_3 = 2, F_4 = 3, F_5 = 5, F_6 = 8, F_7 = 13, \ldots.$

An estimate for Fibonacci numbers

Show that for any natural number $n \geq 4$, we have

$$\left(\frac{8}{5}\right)^{n-2} < F_n < \left(\frac{9}{5}\right)^{n-2}.$$

By the way, the golden ratio ϕ is 1.618... and it is between 8/5 and 9/5.

We will see later that $F_n/F_{n-1} \to \phi = 1.618...$ as $n \to \infty$].

An estimate for Fibonacci numbers, contd

Solution:
Check this is true for $n = 4$ and 5.
This is because we will need *two previous steps* to prove the n-th step.

Since by strong induction we assume the statement is true for all $k < n$, it could certainly be assumed true for $n - 1$ and $n - 2$.

So we will assume that $(8/5)^{n-3} < F_{n-1} < (9/5)^{n-3}$ and $(8/5)^{n-4} < F_{n-2} < (9/5)^{n-4}$ and try to prove it for n.

An estimate for Fibonacci numbers, contd

$$\left(\frac{8}{5}\right)^{n-3} + \left(\frac{8}{5}\right)^{n-4} = \left(\frac{8}{5}\right)^{n-4}\left(\frac{8}{5}+1\right) < F_n = F_{n-1} + F_{n-2}$$

$$< \left(\frac{9}{5}\right)^{n-3} + \left(\frac{9}{5}\right)^{n-4} = \left(\frac{9}{5}\right)^{n-4}\left(\frac{9}{5}+1\right)$$

Now $(8/5)+1 = 13/5$ and $(13/5) > (8/5)^2 = 64/25$ because $13/5 = 65/25$ (multiply above and below by 5). Therefore in the last inequality we can replace 13/5 by $(8/5)^2$.
Similarly $(9/5)+1 = 14/5$ and $(14/5) < (9/5)^2 = 81/25$ because $14/5 = 70/25$ (multiply above and below by 5). Therefore in the last inequality we can replace 14/5 by $(9/5)^2$.

An estimate for Fibonacci numbers, contd

We get

$$\left(\frac{8}{5}\right)^{n-2} = \left(\frac{8}{5}\right)^{n-4}\left(\frac{8}{5}\right)^{2} < F_n$$

$$\textit{and} \quad F_n < \left(\frac{9}{5}\right)^{n-4}\left(\frac{9}{5}+1\right) < \left(\frac{9}{5}\right)^{n-4}\left(\frac{9}{5}\right)^{2} = \left(\frac{9}{5}\right)^{n-2}$$

Thus we have proved the statement for n and the proof is complete.

Proof by strong induction – exercises 1-4.

1. Show that every integer greater than 1 can be broken down into product of prime numbers, using strong induction.
2. Use strong induction to prove that postage of 4 cents or more can be obtained by using only 2 cent and 5 cent stamps.
3. Show that postage of 24 cents or more can be obtained by using only 5 cent and 7 cent stamps.
4. The sequence $g_1, g_2, ...$ is defined by the recurrence relation $g_n = g_{n-1} + g_{n-2} + 1, n \geq 3$, and initial conditions $g_1 = 1, g_2 = 3$. By using mathematical induction or otherwise, show that $g_n = 2F_{n+1} - 1, n \geq 1$, where $F_1, F_2, ...$ is the Fibonacci sequence $1, 1, 2, 3, ...$

Strong induction – exercise 1 solution.

1. **Base case:** Enough to check for 2. 2 is a prime so it is trivially a product of primes.
Assumption: Assume true for all k such that $1 < k < n$. In other words, any such k can be broken down into product of primes.
Proving for n using numbers smaller than n : Then for n it is either a prime or it has a factor m other than 1 and itself.
So let $n = md$ where $1 < m < n$. Then d is strictly between 1 and n as well, because otherwise m would have to be 1 or n.
So *both m and d* satisfy the condition that they lie between 1 and n and so the assumption applies for them. Thus m and d are products of some primes.
If $m = p_1 p_2 ... p_k$ and $d = q_1 q_2 ... q_r$ then
$n = md = (p_1 p_2 ... p_k)(q_1 q_2 ... q_r)$ is also a product of primes.
This concludes the proof.

Strong induction – exercise 2 solution.

1. **Base case:** Enough to check for 4 and 5. Reason will become clear below. 4 = 2+2 and 5 trivially a sum of 5's.
Assumption: Assume true for all k such that $1 < k < n$. In other words, any such k can be broken down into sums of 2's and / or 5's.
Proving for n using numbers smaller than n : We can go from $n - 2$ to n. Since we already proved for 4 and 5, enough to start with $n = 6$.
$n - 2$ satisfies the condition that it lies 4 and n and so the assumption applies for $n - 2$.
So $n - 2$ is a sum of 2's and / or 5's.
But then $n = (n - 2) + 2$ so it should also be a sum of 2's and / or 5's.
This concludes the proof.

Proof by induction – exercises 5,6.

1. Suppose a sequence a_n is defined by $a_1 = 1, a_2 = 2, a_3 = \frac{1+2}{2} = \frac{3}{2}, a_4 = \frac{2+(3/2)}{2} = \frac{7}{4}, ..., a_n = \dfrac{a_{n-1} + a_{n-2}}{2}$.... Show that $a_n < 2$ for all $n > 2$ using strong induction.
2. Use basic induction to prove that any positive integer leaves a remainder of 0,1 or 2 when divided by 3.

Solution to Exercise 3 – base cases

Here the base cases are 24,25,26,27,28.

Reason: The idea for proving it using strong induction is that in order to get a number n as a combination of 5 and 7, we use some number below n that is already as a combination of 5 and 7. If we can add 5 and get n then n is also such a combination. So $n - 5$ is the closest such number. (Can also use 7 but then you would have to go even further back, to $n - 7$).
But you cannot do this if $n - 5$ is smaller than 24 because not all numbers smaller than 24 are not combinations of 5 and 7. For instance but $28 - 5 = 23$ and 23 is not a combination of 5 and 7. So 28 is one of the base cases.
In general $n - 5 < 24 \implies n < 29$. So the base cases to be proved are 24,25,26,27,28.

Solution to Exercise 3 -conclusion

$24 = (2 \times 5) + (2 \times 7); 25 = 5 \times 5$; $26 = (3 \times 7) + 5$
$27 = (4 \times 5) + 7$; $28 = 4 \times 7$.

Now we are ready to prove for all n.

Assume every $k < n$ is a combination of 5 and 7. (Actually this is more than we need! We only need $k = n - 5$).

Then $n - 5 = 5x + 7y$ and
$n = (n - 5) + 5 = 5x + 7y + 5 = 5(x + 1) + 7y$.

So assuming for all numbers smaller than n we have proved that it works for n also.

This means we can prove it for 29, 30, 31, and so on for all ensuing natural numbers because all we need to do is to look at the number 5 less than given number and add 5 to it.

Solution for exercise 4

We have
$g_1 = 1 = 2(1) - 1 = 2f_2 - 1, g_2 = 3 = 2(2) - 1 = 2f_3 - 1$, so the statement is true for $n = 1, 2$.
We need only two base cases because in the recursion formula only previous two terms are used. Note that we are using strong induction here.
Assume $g_{n-1} = 2f_n - 1, g_{n-2} = 2f_{n-1} - 1$.
Then $g_n = g_{n-1} + g_{n-2} + 1 = (2f_n - 1) + (2f_{n-1} + 1) + 1 = 2(f_n + f_{n-1}) - 1 = 2f_{n+1} - 1$ because by definition of Fibonacci sequences, $f_{n+1} = f_n + f_{n-1}$.
This is the statement for n and we have proved it assuming the statement true for $n - 1$ and $n - 2$.

Solution for exercise 5

Solution for 5: The base cases are $n = 3$ and 4 because, like the Fibonacci sequence seen earlier, this sequence uses two consecutive numbers to get the third one. Note that statement is not true for $n = 2$ because $a_2 = 2$ is not less than 2. So we have to start with a_3. $a_3 = 3/2 < 2$, $a_4 = 7/4 < 2$, so statement is true for the base cases.
Assuming it true for all $2 < k < n$, in particular for $n - 1$ and $n - 2$, and $n > 2$, we get $a_{n-1} < 2$, $a_{n-2} < 2$ and

$$a_n = \frac{a_{n-1} + a_{n-2}}{2} = \frac{a_{n-1}}{2} + \frac{a_{n-2}}{2} < \frac{2}{2} + \frac{2}{2} = 2.$$

Solution for exercise 6

Solution for 6: Base case is just 0. It leaves a remainder of 0 when divided by 3.
Suppose it is true for n. Then it leaves a remainder of 0,1, or 2 means $n = 3m, 3m + 1$, or $3m + 2$.
Then $n + 1 = 3m + 1, 3m + 2$, or $3m + 3$.
$3m + 1$ and $3m + 2$ leave a remainder of 1 and 2 respectively.
$3m + 3 = 3(m + 1)$ leaves a remainder of 0. Hence we are done.

A more complicated problem

Prove using strong induction that every positive integer n is either a power of 2 or can be written as the sum of distinct powers of 2. In other words, $n = 2^{a_1} + 2^{a_2} + ... + 2^{a_m}$ and $a_1, a_2, ..., a_m$ are all different.
Try a few examples:
$1 = 2^0, 2 = 2^1, 3 = 2^0 + 2^1, 4 = 2^2, 5 = 2^2 + 2^0, ...$
We see that it seems to be true for all the small cases, and we can increase by 1 or by 2 or by $3 = 1 + 2$ and so on.
So the base case can be just 1 and it is a power of 2.
Assume statement is true for all k with $1 \leq k < n$. Need to prove it for n.
If n is itself a power of 2 we are done.

Contd: A more complicated problem

If not, let $2^t < n$ be the largest power of 2 smaller than n. Then $n - 2^t$ is strictly smaller than n because $2^t \geq 1$. Then $n - 2^t = 2^{a_1} + 2^{a_2} + ... + 2^{a_m}$ for distinct a_i by the induction hypothesis.
So $n = 2^t + 2^{a_1} + 2^{a_2} + ... + 2^{a_m}$. Only thing remaining to prove is that t is different from all the a_i. Suppose not. Suppose $a_1 = t$. Then we have $n = 2^t + 2^t + 2^{a_2} + ... + 2^{a_m}$. But $2^t + 2^t = 2^{t+1}$. This means $2^{t+1} \leq n$ because n is 2^{t+1} plus (possibly) some other numbers. This contradicts the fact that 2^t is the largest power of 2 smaller than n.
If $t = a_i$ for any other i proof is similar.