Chapter I. Section 1.

2-5-2018 class notes

LINEAR AND RATIONAL EQUATIONS

Why do we need algebra?

Algebra is just a convenient way to solve complicated problems from real life without having to do it all in your head.

Mainly it is a symbolic way to represent problems involving numerical quantities.

A LITTLE BIT OF HISTORY

Egyptians used some version of Algebra.

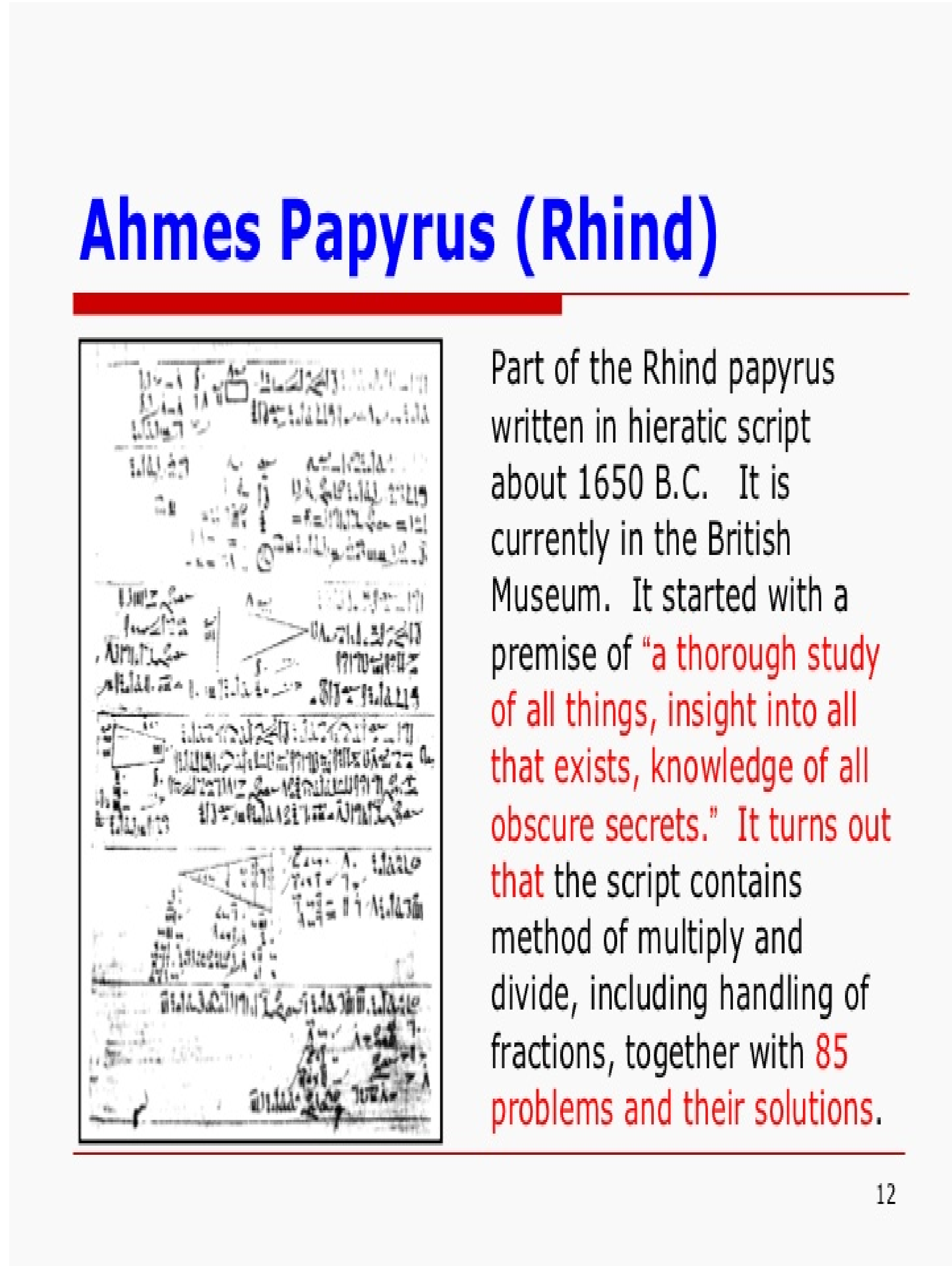
Although they did not have the symbolic language we have today, they managed to solve lots of problems that require manipulations of numbers.

They had to represent everything in words, though.

Greece, China, Babylon and India were some other places with some early form of algebra.

The modern version of Algebra came from Islamic scholars who in turn might have learnt it from the other civilizations.

AN EXAMPLE FROM EGYPT



An example from an Egyptian "math book"

PROBLEM 26, RHYND MATHEMATICAL PAPYRUS

A quantity, $\frac{1}{4}$ of it added to it, becomes 15. What is the quantity?

Today we would write this as

$$x + \frac{1}{4}x = 15$$

And we can solve it easily as follows:

$$x + \frac{1}{4}x = x(1 + \frac{1}{4}) = 15 \implies x = \frac{15}{5/4} = 15 \times \frac{4}{5} = 12.$$

The Egyptians solved it using what is today called "Method of False Position."
For example, to solve x+ (1/4) x = 15, they guess x = 4.
Plugging in 4 into the equation you get then get 4+1 = 5.
Now since 15 = 3 times 5, they guess that real answer must be 12.

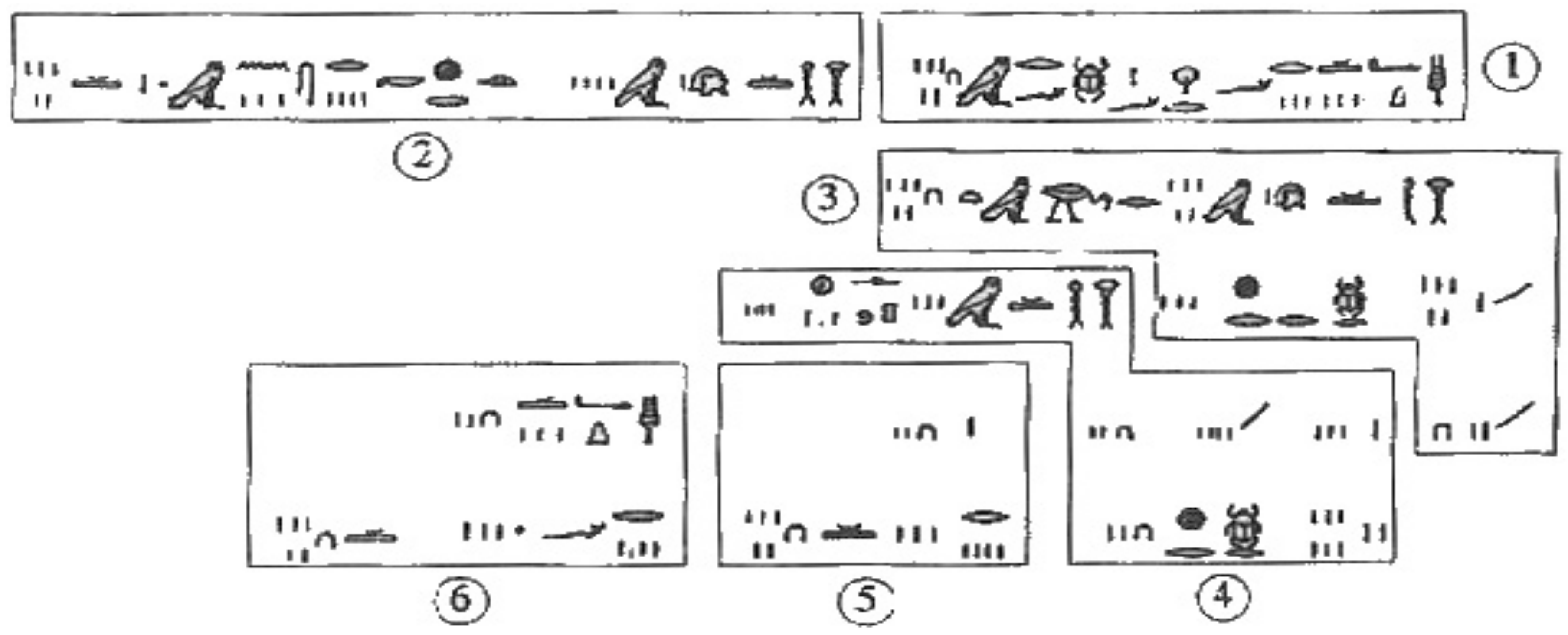


Table 1. Description of Problem 26 of the Rhind Mathematical Papyrus. Adapted from Chabert [8].

Step	Transcription of hieroglyphics	Description using algebraic notation
1	A quantity, $\frac{1}{4}$ of it added to it, becomes 15	$x + \frac{1}{4}x = 15$
2	Operate on 4; make thou $\frac{1}{4}$ of them, namely 1 The total is 5.	Guess $x = 4$ $4 + 1 = 5$.
3	Operate on 5 for the finding of 15 \1 5 \2 10 There becomes 3.	Divide: $\frac{15}{5} = 3$
4	Multiply: 3 times 4. 1 3 2 6, \4 12 There becomes 12.	Multiply wrong answer ($x = 4$) by 3: $3 \times 4 = 12$.
5	1 12, $\frac{1}{4}$ 3 Total 15	$12 + \frac{1}{4}(12) = 15$
6	The quantity is 12. $\frac{1}{4}$ of it is 3; the total is 15.	Thus, $x = 12$.

The first step of the method they used to solve for x in the rhetorical analog of the linear equation (11) was to choose two different initial guesses (false positions) of the solution. There were no restrictions on the initial guesses. Suppose that the first guess of the solution is $x = x_0$; then we get the corresponding residual error e_0, where

$$ax_0 + b - c = e_0. \tag{12}$$

Suppose that the second guess of the solution is $x = x_1$; then we get the corresponding residual error e_1, where

$$ax_1 + b - c = e_1. \tag{13}$$

LINEAR EQUATIONS IN ONE VARIABLE

Equations in a single variable, usually denoted x, of degree 1.

To solve, bring x to one side by adding or multiplying by suitable quantities.

Example:
To rent a storage unit, a customer must pay
a fixed deposit of $150 plus $52.50 in rent each month.
Write a model for the cost C (in $) to rent the unit for t months.
If Winston has $1200 budgeted for storage, for how many months can he rent the unit?

First identify the variable. What is the unknown?
Here t is what we are trying to figure out.

Now **write everything in terms of t**.

$$C = 150 + (52.5) \times t$$

$$So\ 1200 = 150 + 52.5t.$$

Now solve for t by getting it by itself on one side.

$$1200 - 150 = 52.5t \implies 1050 = 52.5t \implies t = \frac{1050}{52.5} = 20.$$

SLIGHTLY MORE COMPLICATED LINEAR EQUATION

$$\frac{y+5}{2} - \frac{y-2}{4} = \frac{y+7}{3} + 1.$$

1. First clear the denominators by multiplying by LCM
2. LCM of 2,3 and 4 is 12. (How is that?)
Get $6(y+5) - 3(y-2) = 4(y+7) + 12$.
SImplifying, get $6y + 30 - 3y + 6 = 4y + 28 + 12$.
3. As before, get the variable y to one side.
Get $6y - 3y - 4y = 12 + 28 - 30 - 6 = 4$.
4. Now get y by itself. Problem solved!
$-y = 4 \implies y = -4$.

5. If you want, check answer in original equation:

$$\frac{-4+5}{2} - \frac{-4-2}{4} = \frac{-4+7}{3} + 1?$$

Yes, because $\frac{1}{2} + \frac{6}{4} = \frac{1}{2} + \frac{3}{2} = 2.$

SOLVING RATIONAL EQUATIONS.
MAIN POINT TO REMEMBER:
ANSWER SHOULD NOT MAKE DENOMINATORS ZERO

$$\frac{2}{x+3} - \frac{5}{x+1} + 2 = 0$$

[Same procedure as before except now denominators are polynomials instead of numbers].

First multiply all by LCM:

$$\frac{2}{x+3} - \frac{5}{x+1} + 2 = 0 \implies (x+1)(x+3)\left(\frac{2}{x+3} - \frac{5}{x+1} + 2\right) = 0$$

The right hand side stays zero because anything times zero is zero.

We get: $2(x+1) - 5(x+3) + 2(x+1)(x+3) = 0 \implies 2x + 2 - 5x - 15 + 2(x^2 + 4x + 3) = 0$

$$\implies 2x^2 + 5x - 7 = 0 \implies (2x+7)(x-1) = 0 \implies x = -7/2 \text{ } or \text{ } 1.$$

Neither of these values makes the denominator of original equation zero.
So both are solutions.
Check your answer in the original equation.

NOTE: SOMETIMES EQUATIONS MAY NOT HAVE SOLUTION!

Example: $x + 1 = x + 2 \implies 1 = 2$ so no value of x will do!

SOME PRACTICE PROBLEMS

ANSWERS AT THE BOTTOM

Simplify the following, as much as you can.

1. In the mid-nineteenth century, explorers used the boiling point of water to estimate altitude. The boiling temperature of water T (in Fahrenheit) can be approximated by the model $T = -1.83a + 212$, where a is the altitude in thousands of feet.

Determine the temperature at which water boils at an altitude of 4000 ft. Round to the nearest degree.

Two campers hiking in Colorado boil water for tea. If the water boils at 193 degrees F, approximate the altitude of the campers. Give the result to the nearest hundred feet.

2. $\dfrac{x-6}{3} + \dfrac{x}{7} = \dfrac{x+1}{3} + 2.$

ANSWERS BELOW

3. $\dfrac{2}{3}y - 5 = \dfrac{1}{6}y - 4.$

5. $\dfrac{5}{2x-3} - \dfrac{3}{5x} = \dfrac{1}{3-x}$

6. $\dfrac{4}{x^2-2x-8} - \dfrac{1}{x^2-16} = \dfrac{2}{x^2+6x+8}$

ANSWERS BELOW

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a = 4 then T = −1.83×4 + 212 = 212 − 7.32
= 204.68

T = 193 = −1.83a + 212 ⇒ 193 − 212 = −1.83a
⇒ −19 = −1.83a ⇒ a = −19/−1.83 = 10.38
Altitude = 10380 feet

2. $\frac{x-6}{3} + \frac{x}{7} = \frac{x+1}{3} + 2.$

First find LCD of 3 & 7 = 21
Multiply all by 21
7(x−6) + 3x = 7(x+1) + 2×21
7x + 3x − 7x = 42 + 7 + 42 = 91
⇒ 3x = 91 ⇒ x = 30.3 approximately
because 91/3 = 30 + 1/3 = 30 + .3333....
= 30.$\overline{3}$
(3 repeats forever) (check by plugging 91/3 in equation)

3. $\frac{2}{3}y - 5 =$ ~~$frac16y - 4$~~. 1/6 y − 4

LCD = 6
Multiplying, 4y − 30 = y − 24
⇒ 4y − y = 30 − 24 ⇒ 3y = 6 ⇒ y = 2
Checks out

6

ANSWER SHOULD NOT EQUAL $\frac{3}{2}, 0, 3$

5. $\frac{5}{2x-3} - \frac{3}{5x} = \frac{1}{3-x}$

First find LCD (LCM of $2x-3$, $5x$ & $3-x$) — Just their product

Multiply all by LCD. Get

$5(5x)(3-x) - 3(2x-3)(3-x) = 1(2x-3)(5x)$

$\Rightarrow 25(3x - x^2) - 3(-2x^2+9x-9) = 10x^2 - 15x$

$\Rightarrow 75x - 25x^2 + 6x^2 - 27x + 27 = 10x^2 - 15x$

$\Rightarrow -25x^2 - 10x^2 + 6x^2 + 75x - 27x + 15x + 27 = 0$

$\Rightarrow -29x^2 + 63x + 27 = 0$

Cannot be factored using whole number expressions

6. $\frac{4}{x^2-2x-8} - \frac{1}{x^2-16} = \frac{2}{x^2+6x+8}$

First find LCD

First factor denominators

$x^2-2x-8 = (x-4)(x+2)$, $x^2-16 = (x-4)(x+4)$, $x^2+6x+8 = (x+4)(x+2)$

LCD $= (x-4)(x+2)(x+4)$

Multiply all by LCD

$4(x+4) - 1(x+2) = 2(x-4)$

$\Rightarrow 4x + 16 - x - 2 = 2x - 8$

$\Rightarrow 3x - 2x = -16 + 2 - 8 \Rightarrow x = -22$

NOT EQUAL TO $4, -2$ or -4

So -22 will not make denominator

Check if -22 works!