R1 Sets and Real Number Line

1-20-2018 class notes

Proof that $\sqrt{2}$ is not a rational number

NOTE: While this is only for your interest, along the way you will see many topics that are discussed in the section R1. Among them are:

1. Writing a fraction as a decimal and vice versa.
2. Decimal expansions of rationals versus those of irrationals.
3. Simplifying and expanding products of powers of numbers.

$\sqrt{2}$ is the length of the hypotenuse of a square with sides of length 1.
So first thing we note that it is *bigger than 1.*
Then we see that it is between 1 and 2 because $1^2 = 1 < 2 < 2^2 = 4$.
Then we checked that it is actually less than 3/2 because $2 < (3/2)^2 = 9/4 = 2.25$.
Similarly $\sqrt{2} > 1.4$ because $2 > 1.4^2 = 1.98$.
We can continue this approach to get as precise an *approximation* as possible. It goes like 1.414...
Note: there is an easier way to find the decimal expansion of a square root, and we will talk about it later if time permits.
All irrational numbers have $non-terminating, non-repeating$ decimal approximation.
If it were rational, decimal expansion would *either terminate or repeat.*
For example, $1/7 = 0.142857142857142857.....$ The block 142857 keeps repeating. This is usually written as $0.\overline{142857}$.

So if we can show that $\sqrt{2}$ *cannot be rational* then we also know decimal expansion of $\sqrt{2}$ is non-terminating.

Suppose it were. Then we have $\sqrt{2} = A/B$ where A, B are natural numbers. Using the fact that every natural number is a product of primes in exactly one way, we write A and B as products of primes $p_1, p_2, ...q_1, q_2..$ etc.,

$$\sqrt{2} = \frac{p_1p_2...p_n}{q_1...q_m} \implies 2 = \frac{(p_1p_2...p_n)^2}{(q_1...q_m)^2} \implies 2 = \frac{p_1^2p_2^2...p_n^2}{q_1^2q_2^2...q_m^2}$$

$$\implies 2q_1^2q_2^2...q_m^2 = p_1^2p_2^2...p_n^2$$

In the last line if 2 is one of the primes $p_1, p_2, ..., q_1, q_2, ...$etc., then after cancellation if necessary we can write it as $2(2^{2m}r_1^2r_2^2...r_k^2) = s_1^2s_2^2...s_l^2$ where $r_1, r_2, ...s_1, s_2...$ are all prime numbers different from 2. Here m could be negative, for example if more of the primes on the right are 2, then when we bring them over to left side you get a negative power of 2. m is really equal to *(number of q_i that equal 2) - (number of p_i that equal 2)*. You get $2m$ in the power because we are squaring all the p_i and q_i.

But now we have an impossible situation because on the left side we have a power of 2 but on the right side we don't have any power of 2. But as mentioned above, you can write a natural number as a product of primes in one way only. Both left and right side equal the same natural number but the prime factors in them are different! This means the assumption we started with, namely that $\sqrt{2}$ is a rational number, must be false.